Lecture notes 7: Nuclear reactions in solar/stellar interiors

Atomic Nuclei

We will henceforth often write protons ${}^1_1\mathrm{p}$ as ${}^1_1\mathrm{H}$ to underline that hydrogen, deuterium and tritium are chemically similar. Deuterium (${}^2_1\mathrm{H}$) consists of a proton and a neutron, tritium (${}^3_1\mathrm{H}$) consists of *two* neutrons and a proton. The mass of deuterium is $< m_\mathrm{p} + m_\mathrm{n}$, the difference between these masses (times c^2) is defined as the binding energy of the nucleus.

Why doesn't the neutron in the deuterium nucleus, or any other nucleus, decay to a proton in the same way a free neutron does? In the case of deuterium it is because the lowest energy state available to a potential two proton atom is positive. *I.e.* it is not bound and a deuterium ion would require energy supplied from elsewhere in order to perform the conversion. (The same is true for a potential two neutron nucleus.)

For the more general case neutron decay may also be hindered by the Pauli exclusion principle which hinders more than two fermions to occupy the same quantum state. Protons and neutrons will each constitute an energy stack in a nucleus: if the nucleus is proton rich rich proton decay can happen since there is "free" slot in the neutron stack at lower energy. Likewise in a neutron rich nucleus neutron decay can happen to occupy an open proton slot in the proton stack. These processes will cause nuclei to tend towards equal numbers of neutrons and protons. Note that since protons are charged they feel electric repulsion from each other and that this slightly increases the spacing of energy levels in the proton stack relative the neutron stack. This will cause very heavy nuclei to tend towards more neutrons than protons.

An example of such a decay is

$${}^3_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + {}^{\;\;0}_{-1}\mathrm{e} + {}^0_0\bar{\nu}$$

which has a half life of 12.5 yr. Other "famous" β-decay processes are

$${}^{13}_{7}\mathrm{N} \rightarrow {}^{13}_{6}\mathrm{C} + {}^0_1\bar{\mathrm{e}} + {}^0_0\nu$$

and

$${}^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} + {}^{\;\;0}_{-1}\mathrm{e} + {}^0_0\bar{\nu}.$$

The latter has a half life of 5 900 yr, a fact that is used in the carbon dating of biological material.

Thermonuclear reactions

On the Sun the most important energy generation process is the **pp** (for proton-proton) chain. The major branch of this chain goes as follows:

$$\begin{aligned}
{}^1_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^2_1\mathrm{H} + {}^0_1\bar{\mathrm{e}} + {}^0_0\nu \\
{}^2_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^3_2\mathrm{He} + {}^0_0\gamma \\
{}^3_2\mathrm{He} + {}^3_2\mathrm{He} &\rightarrow {}^4_2\mathrm{He} + 2\times{}^1_1\mathrm{H}
\end{aligned}$$

Some comments are worth noting on these processes:

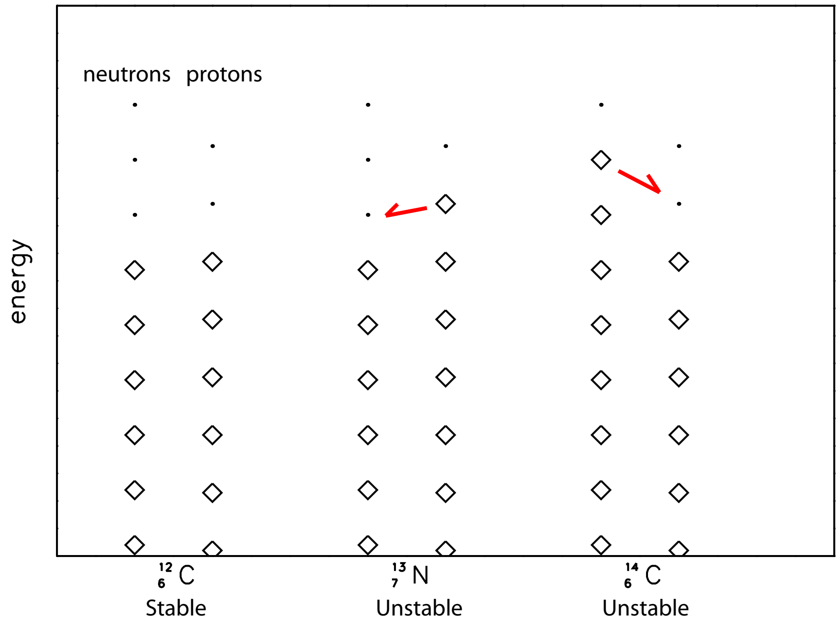

Figure 1: Structure of various carbon and nitrogen nuclei showing an illustration of the neutron and proton energy stacks. Both types of particles are fermions and therefore are bound by Pauli's exclusion principle, a nuclei is unstable and β-decay can occur when there is an 'open' position in the adjacent stack.

1. The first link in this chain involves the weak force and is as such very slow, particularly as the weak reaction must happen during the short time two protons are flying past each other.

2. The second link in the chain has two particles combining to $^3_2$He, a photon must also be emitted since two particles in general cannot combine to one while still conserving energy **and** momentum. The required particle must have 0 charge, 0 baryon number and 0 lepton number, *i.e.* a photon.

3. It might be thought that the process

$$^3_2\mathrm{He} +^1_1 \mathrm{H} \rightarrow^4_2 \mathrm{He} +^0_1 \bar{\mathrm{e}} +^0_0 \nu$$

   would be more common than the third link in the chain since there is *much* more hydrogen than $^3_2$He. This is, however, not the case: the process shown here involves the weak force and is as such very slow.

4. In sum the pp chain involves combining six protons to a $^4_2$He and two protons, or net converting four protons to $^4_2$He. And we remind the reader that the mass of the helium atom is $3.97m_\mathrm{p}$ giving an efficiency of 0.7%.

The CNO cycle

Since the weak (and therefore slow) force is involved in the pp-chain, the temperature of the solar core is warmer than it would otherwise be in order to fuse hydrogen to helium. This led Hans Bethe to hypothesize that processes involving catalysts could be more efficient. The CNO-cycle is such a process that works more efficiently than the pp-chain for stars with slightly hotter cores than the Sun. The reason this process is faster is due the fact that the necessary β-decay in the reactions happens inside nitrogen and oxygen nuclei where nucleons are in continual contact and the chances of a decay are therefore much greater than the short time two protons have available while flying past each other. The CNO-cycle is dependent on the β-decay of ${}^{13}_{7}\mathrm{N}$ which has a lifetime of 870 s, and of ${}^{15}_{8}\mathrm{O}$ with a lifetime of 178 s.

$$\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + {}^{0}_{0}\gamma \\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + {}^{0}_{1}\bar{\mathrm{e}} + {}^{0}_{0}\nu \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + {}^{0}_{0}\gamma \\
{}^{14}_{6}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + {}^{0}_{0}\gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + {}^{0}_{1}\bar{\mathrm{e}} + {}^{0}_{0}\nu \\
{}^{15}_{7}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}
\end{aligned}$$

Temperature sensitivity

Recall that at relatively low temperatures material systems prefer more binding energy. In this sense the temperature is low in the solar core and nuclei prefer to fuse — as long as they can over-win the repulsive Coulomb potential.

Recall that the forces between a system of two particles may be written in terms of the **reduced mass** $m = m_1 m_2/(m_1 + m_2)$ such that $F_{12} = mdv/dt$ and that the kinetic energy of the system in the center of mass is

$$E_k = \frac{1}{2}mv^2$$

where v is the relative velocity of the two particles.

The potential energy of two particles that are interacting by the Coulomb force is

$$E_p = -\frac{q_1 q_2}{4\pi\epsilon_0 r}.$$

Thus if the relative velocity is v at infinite separation where $E_p = 0$, the particle can come to within

$$r = \frac{q_1 q_2}{2\pi\epsilon_0 m v^2}.$$

Assume now that a typical nuclear separation is $<< r$. The quantum mechanical probability for **tunneling** through the potential barrier can be shown to be $\exp(-2\pi^2 r/\lambda)$, where λ is the DeBroglie wavelength h/mv at infinite separation. Inserting this gives

$$\exp(-q_1 q_2 \pi/\epsilon_0 v h)$$

for the tunneling prabability.

On the other hand the probability for finding a given speed between v and $v + dv$ is given by the Boltzmann distribution in a perfect gas

$$\exp(-mv^2/2kT).$$

The probability of a nuclear reaction is then given as the product of these two (independent) probabilities.

We may find the maximum by setting the derivative of the product to 0. This gives that reactions are most probable when

$$v^3 = \frac{q_1 q_2 kT\pi}{m\epsilon_0 h}$$

computing the r that corresponds to this v when considering hydrogen reactions and a solar core temperature of $T = 15 \times 10^6$ K gives $r \approx 10^{-13}$ m which shows that our assumption of that $r >>$ than nuclear separations, which typically are 10^{-15} m.

Inserting the most probable speed into our original expression for the probability gives an expression

$$\exp\left[-(T_0/T)^{1/3}\right] \qquad \text{where} \qquad T_0 = (\frac{3}{2})^3(\frac{q_1 q_2 \pi}{\epsilon_0 h})^2(\frac{m}{k}) \tag{1}$$

for hydrogen reactions.

Binding energies of atomic nuclei

The general tendency is for binding energy per nucleons to be greater for heavier nuclei up to ^{56}Fe, where-after it decreases again for the heaviest nuclei. But notice that $^4_2$He has more binding energy than ^{6}Li and the next few atoms in the periodic table.

Explaining the general tendency is relatively straightforward: it is a result of the balance between the attractive and repulsive strong force as modified by the Coulomb force. Atomic nuclei are more or less incompressible since the strong force is attractive at large distances but repulsive at short distances. Thus, nuclei will grow in physical size as they grow in baryon number. This means that the Coulomb interaction becomes steadily more important as distances grow larger compared to the range of the strong force. It is therefore generally advantageous for stars to construct heavier nuclei up to ^{56}Fe after which the binding energy per nucleon decreases due the increased importance of the Coulomb force.

Some configurations of nuclei are especially stable since protons and neutrons actually can stack two at a time with opposed spins in any energy level. Thus nuclei having even numbers of protons and neutrons are more stable than their immediate neighbors; $^4_2$He, $^{12}_6$C, and $^{16}_8$O all form local peaks in nuclear binding energy. This gives rise to the question: How does a star proceed from $^4_2$He in building heavier nuclei?

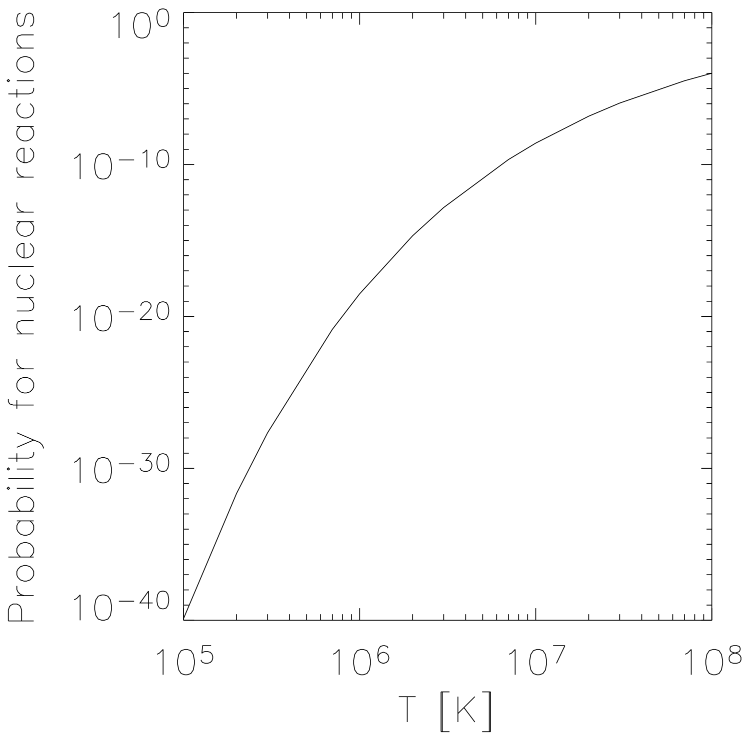

Figure 2: The probability factor (1) $\exp[-(T_0/T)^{1/3}]$ plotted as a function of temperature.

The answer lies in some "luck": The $^8_4$Be formed in the process

$$^4_2\mathrm{He} + ^4_2\mathrm{He} \rightarrow ^8_4\mathrm{Be} +^0_0 \gamma$$

has a lifetime of only 2.6×10^{-16} s. However, at temperatures of $T = 10^8$ K and densities of $\rho = 10^8$ kg/m^3, such as may be found in a stellar core in late stages of its evolution, we find a non-negligible beryllium to helium fraction of $n_{\mathrm{Be}}/n_{\mathrm{He}} = 10^{-9}$. This is a large enough number that the reaction

$$^8_4\mathrm{Be} + ^4_2\mathrm{He} \rightarrow ^{12}_6\mathrm{C}^* +^0_0 \gamma$$

can proceed at a reasonable rate. Note that the carbon atom formed is in an *excited nuclear state*, it is the fact that there exists such a state at the correct resonant energy — as predicted by Fred Hoyle — that gives this reaction a sufficiently large cross section to proceed at a reasonable rate in "reasonable" stellar conditions. (How do we define "reasonable stellar conditions"?)

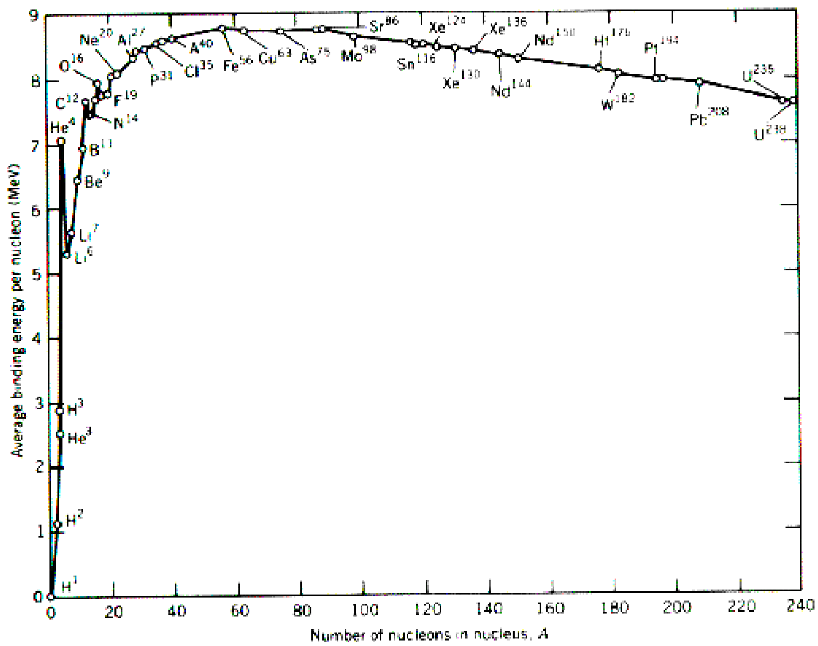

Figure 3: Binding energy per nucleon.

The fusion processes described above constitute the triple-α process, named for the three helium nuclei that go into forming a carbon atom.

General pattern of thermonuclear fusion

The general pattern of thermonuclear fusions proceeds by like chains; for example to form oxygen or magnesium we have the chains

$$ {}^{4}_{2}\mathrm{He} + {}^{12}_{6}\mathrm{C} \to {}^{16}_{8}\mathrm{O} + {}^{0}_{0}\gamma $$

$$ {}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \to {}^{24}_{12}\mathrm{Mg} + {}^{0}_{0}\gamma $$

These chains must have steadily higher temperatures as the charges of nuclei grow in order to overcome Coulomb repulsion.

The r and s processes

There is an exception to this temperature rule if there is a source of neutrons present as neutrons do not feel the Coulomb force. It is possible to distinguish material synthesized in a a neutron rich environment from that synthesized in a neutron poor one. **s-process** ('s' for slow) elements are those formed where β-decay is expected to occur before a neutron is absorbed, while **r-process** ('r' for rapid) elements are those formed where new neutrons can be absorbed readily. Sources of neutrons are various, for examples such chains as

$$^{4}_{2}\mathrm{He} + ^{13}_{6}\mathrm{C} \rightarrow ^{16}_{8}\mathrm{O} + ^{1}_{0}\mathrm{n}$$

$$^{16}_{8}\mathrm{O} + ^{16}_{8}\mathrm{O} \rightarrow ^{31}_{16}\mathrm{S} + ^{1}_{0}\mathrm{n}$$

Free neutrons produced in this manner are a way of forming elements beyond the iron peak in binding energy. In ordinary circumstances in stellar cores it is the s-processes that dominate, in extreme situations such as in supernova r-process nucleosynthesis can occur.

Understanding of abundances can therefore lead to an understanding of the history of our galactic environment. In addition, the radioactive elements formed with various lifetimes gives one a chance of dating various materials the solar system is composed of.

Solar neutrinos

The low opacity of the Sun's material to neutrinos implies that they represent a method of directly observing nuclear fusion in the core of the Sun. However, the neutrinos produced in the first stage of the pp-chain shown above are of fairly low energy — because the energy produced is shared with a positron — and were until recently difficult to detect. Luckily, there are side branches of the pp-chain that produce more energetic neutrinos.

More energetic neutrinos are produced in the so-called **pep-chain**

$$^{1}_{1}\mathrm{H} + ^{0}_{-1}\mathrm{e} + ^{1}_{1}\mathrm{H} \rightarrow ^{2}_{1}\mathrm{H} + ^{0}_{0}\nu$$

This branch proceeds at at rate roughly 1/400 of the main pp-chain since it depends on a *three*-body collision in which two protons and one electron must collide at the same time: a phenomena which is more unlikely than a two-body collision between two protons.

Even less likely, but producing very energetic neutrinos is the following branch of the pp-chain.

$$\begin{array}{rcl}
^{3}_{2}\mathrm{He} + ^{4}_{2}\mathrm{He} & \rightarrow & ^{7}_{4}\mathrm{Be} + ^{0}_{0}\gamma \\
^{7}_{4}\mathrm{Be} + ^{1}_{1}\mathrm{H} & \rightarrow & ^{8}_{5}\mathrm{B} + ^{0}_{0}\gamma \\
^{8}_{5}\mathrm{B} & \rightarrow & ^{8}_{4}\mathrm{Be} + ^{0}_{1}\bar{\mathrm{e}} + ^{0}_{0}\nu \\
^{8}_{4}\mathrm{Be} & \rightarrow & ^{4}_{2}\mathrm{He} + ^{4}_{2}\mathrm{He}
\end{array}$$

These neutrinos are energetic enough to be caught in tanks of cleaning fluid, C_2Cl_4, by the reaction

$$^{37}_{17}\mathrm{Cl} + ^{0}_{0}\nu \rightarrow ^{37}_{18}\mathrm{Ar} + ^{0}_{-1}\mathrm{e}.$$

The argon produced is chemically separated from the system. Left to itself the argon can react with an electron (in this case with its own inner shell electron) by the converse process

$$^{37}_{18}\mathrm{Ar} + ^{0}_{-1}\mathrm{e} \rightarrow ^{37}_{17}\mathrm{Cl} + ^{0}_{0}\nu.$$

The chlorine atom is in an excited electronic state which will spontaneously decay with the emission of a photon. The detection of such photons by a photomultiplier then is an indirect measurement of the solar neutrino flux.

The measured count of neutrinos is consistently lower by a factor of three than that expected produced in the solar core. After many years of study it seems that the resolution of this "solar neutrino problem" is to be found in **neutrino oscillations**, *i.e.* that neutrinos oscillate between the three different flavors; electron-, μ-, and τ-neutrinos.

Gallium detection, the Kamiokande detector and the direct observation of neutrino oscillations.

The Soviet-American Gallium Experiment (SAGE) in the Caucasus, and GALLEX beneath the Italian Alps measure low energy **pp**-chain neutrinos that dominate the Solar neutrino flux via the reaction

$$^{71}_{31}\mathrm{Ga} + ^{0}_{0}\nu \rightarrow ^{71}_{32}\mathrm{Ge} + ^{0}_{-1}\mathrm{e}.$$

The Kamiokande II detector uses water and detects the Cherenkov radiation (light) that is produced when neutrinos scatter electrons, causing electrons to travel faster than the local speed of light in water. The Kamiokande reactor can observe both electron and μ neutrinos. By classifying the neutrino interactions according to the type of neutrino involved (electron-neutrino or μ-neutrino) and counting their relative numbers as a function of the distance from their creation point *e.g.* the Sun, it is concluded that the muon-neutrinos are "oscillating." Oscillation is the changing back and forth of a neutrinos type as it travels through space or matter. This can occur only if the neutrino possesses mass. The Super-Kamiokande result indicates that μ-neutrinos are disappearing into undetected tau-neutrinos or perhaps some other type of neutrino (e.g., sterile-neutrino). (See also `http://www.ps.uci.edu/~ superk/`)

Exercises: Nuclear Reactions

1. (a) Assume that the Sun was initially composed of 70% by mass of hydrogen. How many hydrogen nuclei were there originally in the Sun if its total mass is 2.0×10^{30} kg. What is the total nuclear energy supply $NE/4$ where $E = 0.03 m_p c^2$, if all the hydrogen could be fused into helium.

   (b) What is the lifetime of the Sun's hydrogen burning phase if it can burn 13% of its initial hydrogen content.

2. Calculate the range $h/m_0 c$ and the attractive part of the nuclear strong force if the exchanged π meson has a rest mass $m_0 = 0.15 m_p$, where m_p is the rest mass of the proton. The π meson contains two of the three quarks that make up a proton or an antiproton. Why does it only have 0.15 of the rest mass mass?

3. (a) The encounter of two nuclei of charges q_1 and q_2 and masses m_1 and m_2 can be analyzed in terms of a single particle of reduced mass $m = m_1 m_2/(m_1 + m_2)$ moving at a relative velocity v in a mutually repulsive Coulomb field. Let v be the relative speed at infinite separation, argue that the total energy of the reduced mass is $mv^2/2$.

   (b) Show that according to classical mechanics the two masses cannot approach closer than the distance r, where the electrical potential energy $(1/4\pi\epsilon_0)(q_1 q_2/r)$ equals $mv^2/2$, i.e. the reduced mass encounters a barrier at
$$r = \frac{1}{4\pi\epsilon_0}\frac{2q_1 q_2}{mv^2}. \tag{2}$$

   (c) Quantum mechanics allow the particle to tunnel closer than r, but the probability of reaching nuclear separations (assumed to be small compared to r) decreases exponentially with the ratio of r to the DeBroglie wavelength $\lambda = h/mv$ associated with the momentum mv of the particle at infinity. Gamow found that
$$\begin{aligned}\text{penetration probability} &\propto \exp(-2\pi^2 r/\lambda)\\ &\propto \exp(-\pi q_1 q_2/\epsilon_0 hv).\end{aligned}$$
   The probability for a nuclear reaction is proportional to the penetration probability multiplied by the nuclear cross section once the particle has penetrated to nuclear distances. For a classical gas, the probability of having relative speed v is given by the Maxwell-Boltzmann distribution
$$\text{probability of relative speed } v \propto \exp(-mv^2/2kT). \tag{3}$$
   The net probability of coming within nuclear distances is therefore proportional to the product of two exponentials: one which increases with increasing v, the other which decreases with increasing v. Find the value v where the probability is maximized.

   (d) Compute the corresponding numerical value of r for the proton-proton reaction at $T = 1.5 \times 10^7$ K.

   (e) The rate of nuclear reactions is proportional to the maximum value of the net probability. This maximum value is given by inserting our expression for v into the product of the two exponentials. Do so and plot the resulting probability function for the temperature range $10^5\ \mathrm{K} < T < 10^8\ \mathrm{K}$

4. The opacity κ of most materials for neutrino capture is on the order of $10^{-21}\ \mathrm{m^2/kg}$. Calculate the mean free path $l = 1/\rho\kappa$ for neutrino capture at the center of the Sun where $\rho \approx 10^5\ \mathrm{kg/m^3}$. Comment on the scale of your answer.