AST1100 Lecture Notes

25-26: Cosmology: nucleosynthesis and inflation

1 A brief history of the universe

1. The temperature and energies in the very early universe ($t < 10^{-30}$s) were so high that we do not know the physics of this era. Some theories suggest that the electromagnetic, strong and weak interactions were all unified at these energies in the so-called GUT (Grand Unified Theory). In this unified theory particles which only exist at huge temperatures and which have never been observed may have dominated the universe. We expect processes like
$$^0_0\gamma +^0_0 \gamma \rightleftharpoons X + \bar{X},$$
where X are yet unknown particles existing in the high energy theories. Due to an asymmetry in these theories, there were more particles than anti-particles present in the universe. The universe was filled with a hot dense plasma of particles which are continiously decaying and created.

2. At a time $\sim 10^{-36}$s after the Big Bang a phase transition in the vacuum may have caused the universe to expand exponentially for a period of about 10^{-34}s. This very rapid expansion is called *inflation.*

3. At the end of inflation in the so-called re-heating period, vacuum energy produced huge amounts of matter and radiation.

4. As the universe expanded and cooled, the temperatures became sufficiently low for the forces and laws of physics to follow currently known theories. The unknown high energy particles decayed to known particles, baryons, mesons, leptons, photons and their antiparticles. Particles were created and destroyed continously in different processes.

5. At a temperature of about 10^{12}K about 10^{-4}s to 10^{-2}s after the Big Bang, the process
$$^0_0\gamma +^0_0 \gamma \rightleftharpoons \mu^+ + \mu^-$$

ceased. The energy of the photons was not high enough to produce the μ leptons and the μ leptons disappeared from the mixture by the annhilation process (left arrow in this equation).

6. About 10^{-2}s to 1s after the Big Bang, the temperature of the universe was $T10^{10}$K to 10^{11}K. Up to this point, the neutrinos were interacting with the other particles in the plasma. At this temperature the interaction probability of the neutrinos has decreased to a level where the neutrinos can be considered to be completely decoupled from the rest of the plasma. From this period, less than one seconds after the Big Bag, the neutrinos could travel freely without beeing scattered on other particles. The neutrinos which decoupled in this period are still present today as the *neutrino background*. If these neutrinos could be observed, a huge amount of information about the universe a very short time after the Big Bang could be obtained. Unfortunately, the neutrinos hardly interact with normal matter at all and detection is very difficult.

7. During the first three minutes after the Big Bang with a temperature of $T = 10^9$K to $T = 10^{10}$K, the process

$${}^0_0\gamma + {}^0_0\gamma \rightleftharpoons {}^{\;\,0}_{-1}\mathrm{e} + {}^0_1\bar{\mathrm{e}}$$

ceases and the positrons disappear from the primordial plasma leaving only a small amount of electrons.

8. When the temperature of the universe reached 10^9K, the protons and neutron combined to form the first atomic nuclei. This era is called the *nucelosynthesis era*.

9. About 50000 years after the Big Bang, the density of matter was now higher than the density of radiation. The universe had reached the epoch of *matter-radiation equality*.

10. About 360000 years after the Big Bang the temperature ($T \approx 3000$K) was sufficiently low for the electrons to combine with the atomic nuclei and form the first neutral atoms in the universe. This era is called *recombination*. Before recombination, the photons were continously scattered on the electrons and photons. The collision cross section for photons on neutral atoms is much smaller than for collisions with charged particles. For this reason, the photons could now free-stream

without being scattered. The photons were decoupled from the matter. These photons which were released from the plasma at this period have free-streamed until today and can be observed as the *cosmic microwave background*. The redshift of the recombination period is $z \approx 1100$. We have learned that

$$\frac{R(t_0)}{R(t)} = 1 + z,$$

meaning that the universe has expanded with a factor of 1100 since the recombination period. The wavelength of the photons emitted at recombination has therefore been stretched with a factor of roughly 1100. The mean energy of photons in a photon gas is $< E_\gamma >= kT$ (can be shown in thermodynamics in the same way as you showed that the mean energy of a particle in an ideal gas is $(3/2)kT$) where T is the temperature of the gas. We know that the energy of the photons decrease as $E' = E/(1+z)$. Using that $E = kT$ we have that $T' = T/(1+z)$. Thus the temperature of the photon gas decreases with a factor 1100. The temperature of the photons in the cosmic background radiation is hence $T = 3000\text{K}/1100 \approx 2.7\text{K}$. We observe the cosmic microwave background as radiation from a black body with a temperature of $T = 2.7\text{K}$ whereas the real temperature of the black body (the primoridal plasma) when the photons were emitted was $T \approx 3000\text{K}$.

11. About $10^8 - 10^9$ years (redshift $6 < z < 20$) after the universe became neutral in the recombination process, gravitational collapse had created the first stars in the universe, the population III stars. The energetic radiation from these stars and other objects like quasars which were formed in this epoch ionized the neutral hydrogen gas. The universe became reionized in the *reionization epoch*.

We will now discuss some of these epochs in some detail.

2 Nucleosynthesis

When the temperature of the primordial plasma was $T \sim 10^{12}\text{K}$ the plasma consisted of photons, electrons, positrons, neutrinos and their antiparticles. There was also a small number of protons and neutrons. The neutrons and protons were continously converted into each other in processes like

$$ {}_0^1\mathrm{n} \rightleftharpoons {}_1^1\,\mathrm{p} + {}_{-1}^{\;0}\,\mathrm{e} + {}_0^0\,\bar{\nu}_e. $$

In statistical physics you will learn about the Boltzmann equation giving the ratio between the abundance of two species in equilibrium. For neutron and protons participating in the above process this equation can be written

$$ \frac{n_n}{n_p} = e^{-(m_n - m_p)/kT}. \tag{1} $$

Here n_n and n_p are the number densities (per real space volume) of neutrons and protons. The exponential contains the energy difference between the two states, the proton and the neutron state, divided by the thermal energy kT. The neutron rest energy is slightly larger than the proton rest energy. Looking at the equation we see that this leads to a smaller number of neutrons than protons. We see from the equation that at high temperatures, the reaction rate for this process is equal in both directions: the high temperature makes the term in the expotential very small and therefore $n_n/n_p \approx 1$. When the universe expands and cools, this changes. The mass difference between the neutrons and protons start to be significant and the ratio decreases.

At a temperature of roughly 10^{10}K (you will show this in the exercises) the reverse reaction, protons and electrons colliding and creating neutrons, stopped. At this temperature, the energy of the neutrino was so small that its probability for taking part in reactions had decreased considerably. At the same time, the thermal energy of the photons had become so small that it could not anymore create electon-positron pairs by the reaction

$$ {}_0^0\gamma \rightleftharpoons {}_{-1}^{\;0}\,\mathrm{e} + {}_1^0\,\bar{\mathrm{e}}. $$

As a result all positrons reacted with electrons and created photon, but the photons were on average not sufficiently energic for the reverse process. The positrons disappeared from the plasma and a small number of electrons remained. Thus, there were suddenly much less neutronins and electrons available for the above process to create neutrons from protons and electrons. The reaction now went only in one direction

$$ {}_0^1\mathrm{n} \rightarrow {}_1^1\,\mathrm{p} + {}_{-1}^{\;0}\,\mathrm{e} + {}_0^0\,\bar{\nu}_e. $$

Inserting $T = 10^{10}$K in equation 1, we find $n_n/n_p = 0.223$. There were 223 neutrons for every 1000 protons. From this point on, the reaction only went in one direction: reducing the number of neutrons in the neutron disintegration process. This reaction could continue until the temperature in the plasma was sufficiently low to allow atomic nuclei to be created. Neutrons bound in atomic nuclei are much more stable.

At a temperature of $T \approx 10^9$K, the temperature was sufficiently low for the reactions

$$
\begin{gathered}
{}_1^1\mathrm{p} +{}_0^1 \mathrm{n} \rightleftharpoons {}_1^2 \mathrm{H} +{}_0^0 \gamma, \\
2{}_1^2\mathrm{H} \rightleftharpoons {}_1^3 \mathrm{H} +{}_1^1 \mathrm{p}, \\
{}_1^3\mathrm{H} +{}_1^2 \mathrm{H} \rightleftharpoons {}_2^4 \mathrm{He} +{}_0^1 \mathrm{n}.
\end{gathered}
$$

The first helium atoms were formed. It took about 187 seconds (you will show this in the exercises) from the temperature had decreased from 10^{10}K to 10^9K. During this time the number of neutrons had been reduced due to neutron decay and the number of protons had increased due to the same process. This ratio was now about 362 neutrons to 2084 protons. Each helium atom has two neutrons so 181 helium atoms were made from the 362 neutrons and 362 protons. Thus the helium to hydrogen ratio was now 181 helium nuceli to 2084-362=1722 hydrogen nuclei (protons). After all neutrons were bound in helium nuclei, the mass fraction of helium in the universe could be written

$$
X_{He} = \frac{4 \times 181}{1722 + 4 \times 181} \approx 0.296.
$$

A more detailed calculation gives 0.24 which is exactly the observed ratio between hydrogen and helium in the universe today. This is one of the most important confirmations of the Big Bang theory, the theory that the universe in its early phase was very hot and very dense.

3 The flatness problem and the horizon problem

Observations have shown that the universe is flat to a very high precision. The WMAP (Wilkinson Microwave Anisotropy Probe) satellite has observed the temperature fluctuations in the cosmic microwave background with high resolution and sensitivity. Using these measurements one finds that $\Omega_0 = 1.01 \pm 0.01$. Even if Ω_0 is not exactly one today, it must have been very close to one in the beginning. We can see this by looking at the first Friedmann equation. Using that

$$\rho(t) = \Omega(t)\rho_c(t) = \Omega(t)\frac{3H^2(t)}{8\pi G}$$

the first Friedmann equation can be written

$$\dot{R}^2(t) - \Omega(t)H^2(t)R^2(t) = -k$$

Dividing by $R^2(t)$ on both sides and using the definition of $H(t)$ we get

$$(\Omega(t) - 1) = \frac{k}{R^2(t)H^2(t)} \tag{2}$$

If we use the results from a matter dominated universe $R(t) \propto t^{2/3}$ giving $H(t) = t^{-1}$ we get

$$(\Omega(t) - 1) \propto t^{2/3}.$$

When $t \to 0$ we see that $\Omega(t) \to 1$. A similar result is found using $R(t)$ for a radiation dominated universe. We see that even if the universe is not completely flat today it must have been very close to flat in early epochs. The question then is why the universe is flat or very close to flat. Is there a physical mechanism which makes it flat? This is called the *flatness problem* in cosmology.

Another problem is the *horizon problem*. The *horizon* at a time t is the distance that a photon could have travelled from the Big Bang $t = 0$ and to time t. Thus, two points in space which are separated by a distance larger than the horizon have never been in causual contact: since a photon could not have managed to go from one point to the other during the life time of the universe, no signal could have communicated any information between these two points. Today we cannot se further out in the universe than to the horizon since photons emitted at the Big Bang from a place further away than the horizon would not have had time to reach us today. But as time goes, the horizon grows. We will find the proper distance to the horizon by considering a photon emitted at $r = 0$ at $t = 0$ and find the distance that the photon has reached at time t. The proper distance to the horizon is found by setting $\Delta t = 0$ in the FRW line element. We also set $\Delta\phi = \Delta\theta = 0$ since the photons moves radially away from $r = 0$. Thus we have for the proper distance d_h to the horizon

$$d_h = R(t)\int_0^r \frac{dr'}{\sqrt{1 - k(r')^2}}.$$

We found in the previous lectures that by using $ds = 0$ in the FRW-metric for a photon we can write

$$d_h(t) = R(t)\int_0^r \frac{dr'}{\sqrt{1-k(r')^2}} = R(t)\int_0^t \frac{dt'}{R(t')}.$$

If we consider an epoch in the matter dominated area, we use that $R(t) \propto t^{2/3}$ giving $d_h(t) = 3t$. Thus the size of the observable universe today is roughly three times the age of the universe, about $13.2 \times 3 \approx 40$Gly (1Gly= 10^9ly).

At recombination, the photons were decoupled from matter and allowed to travel freely. In figure 1 we see photons from the recombination era ($r = r_{\rm rec}$ and $t = t_{\rm rec}$) streaming to us ($r = 0$) and received at present time $t = t_0$. In the figure I have also indicated the size of the horizon in the recombination era ($t = t_{\rm rec}$). Point B is at a distance equal to the horizon at $t = t_{\rm rec}$ away from point A. We observe the points A and B in the cosmic microwave background at an angular distance $\Delta\theta_h$ in the sky. Two points with an angular distance larger than $\Delta\theta_h$ have never been in causual contact. We will try to calculate this angle $\Delta\theta_h$ in order to find how far away points that we observe in the background radiation can be and still have been in causual contact.

Before we start to calculate the angular extension $\Delta\theta_h$ of the horizon at recombination, we will try to find the angular extension $\Delta\theta$ of an object at coordinate distance r (using $r = 0$ for the Earth) and proper extension D taken at the time when light was emitted from the object. The situation is depicted in figure 2. We remember from the theory of relativity that the proper length of an object, which in this case is D, is found by measuring the the distance between the end points at the same time, $\Delta t = 0$. This time t is the moment when light was emitted from both ends of the object. The end points A and B of the object are situated at the same position r such that $\Delta r = 0$ and we orient it such that also the ϕ coordinate of A and B are the same such that $\Delta\phi = 0$. The FRW line element then gives

$$D = \Delta s = rR(t)\Delta\theta = \frac{r\Delta\theta}{1+z},$$

where z is the redshift of the object. We have set $R_0 = 1$ and used the expression $R_0/R(t) = 1 + z$ from the previous lectures. We are now in the position to find the angular extension $\Delta\theta_h$ of the horizon at recombination. We found above that the proper size of the horizon is $D = 3t$. We thus have

$$\Delta\theta_h = \frac{(1+z)3t_{\rm rec}}{r_{\rm rec}} \tag{3}$$

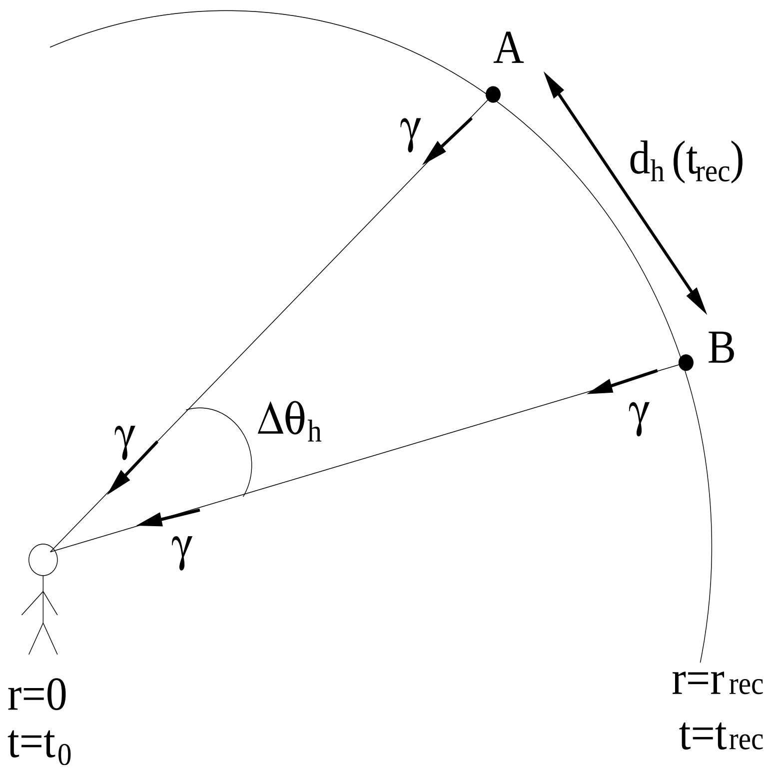

Figure 1: Photons emitted at two points A and B in the recombination epoch. The points A and B were seperated by the size of the horizon d_h at the time of recombinations. Points separated by larger distances had thus never been in causal contact before recombination. We should therefore not observe any correlation between the temperature of cosmic background radiation photons emitted at points separated by distances larger than the distance between A and B

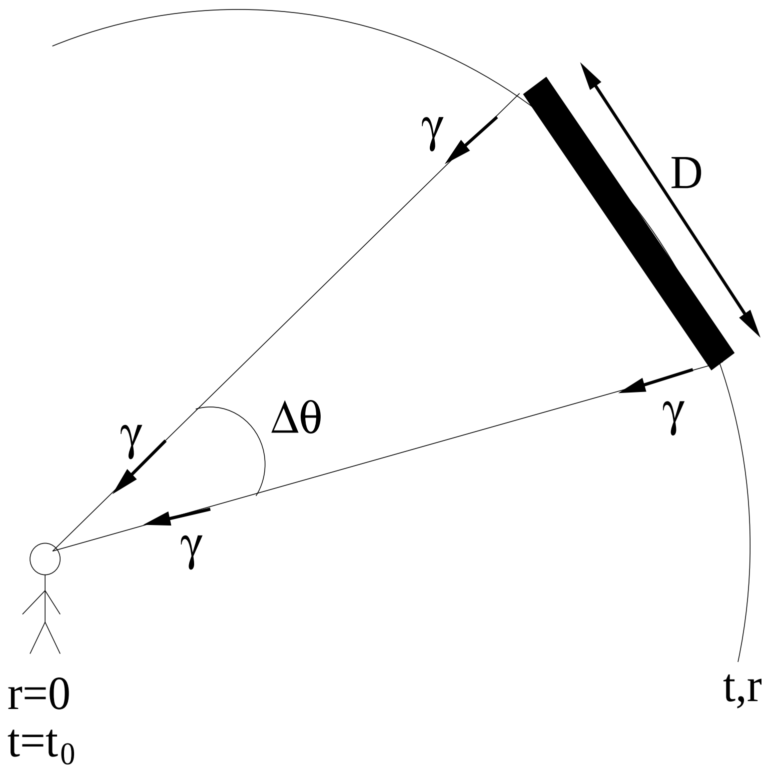

Figure 2: Photons emitted at the two points of a galaxy with proper distance D at the time of emission.

In order to be able to insert numbers here, we need the coordinate distance $r_{\rm rec}$ of recombination. To find the coordinate distance we will follow a photon emitted at recombination. In a flat universe $k = 0$, we have from the FRW-metric that

$$r_{\rm rec} = -\int_{r_{\rm rec}}^{0} dr' = -\int_{t_{\rm rec}}^{t_0} \frac{dt'}{R(t')},$$

where we again used that $\Delta s = 0$ for light and that the light moves radially $\Delta\phi = \Delta\theta = 0$. Since recombination, the universe was matter dominated and we therefore have $R(t) = (t/t_0)^{2/3}$ giving

$$r_{\rm rec} = \int_{t_{\rm rec}}^{t_0} dt' \left(\frac{t_0}{t'}\right)^{2/3} = 3t_0 \left[1 - \left(\frac{t_{\rm rec}}{t_0}\right)^{1/3}\right].$$

Inserting this in equation 3 we have

$$\Delta\theta_h = \frac{(1+z)t_{\rm rec}}{t_0} \left[1 - \left(\frac{t_{\rm rec}}{t_0}\right)^{1/3}\right]^{-1}.$$

Inserting the redshift of recombination $z_{\rm rec} \approx 1100$, the current age of the universe $t_0 \approx 13.2 \times 10^9$ years and the time of recombination $t_{\rm rec} \approx 360000$ years, we have $\Delta\theta_h \approx 1.8^\circ$. The real angular extension of the horizon at recombination is smaller since no photon can move directly in a straight line from A to B without being scattered on other particles. One should really use the sound speed in the plasma which is roughly $\sqrt{3}$ times smaller than the speed of light, such that $D = \sqrt{3}t_{\rm rec}$. Dividing by this correction factor of $\sqrt{3}$ we obtain a result very close to the correct number $\Delta\theta \approx 1^\circ$. Thus, if we observe the microwave background at two points in the sky separated by more than about one degree, these points were never in contact with each other, no information can have travelled from one point to the other.

In figure 3 we see the latest obsevation of the cosmic microwave background from the WMAP satellite. We see the temperature of the background radiation measured in different directions on the sky. The mean temperature of $T = 2.7126$K has been subtracted such that we only see the fluctuations around the mean temperature. These fluctuation come from the density fluctuations in the plasma at recombination. Many of the fluctuations you see in figure 3 have an angular extension of several degrees. This means that the density fluctuations at recombinations, the lumps in the plasma, were larger than the horizon. But how can lumps which are larger than the horizon form? In order to create lumps in the plasma, the two end points must have

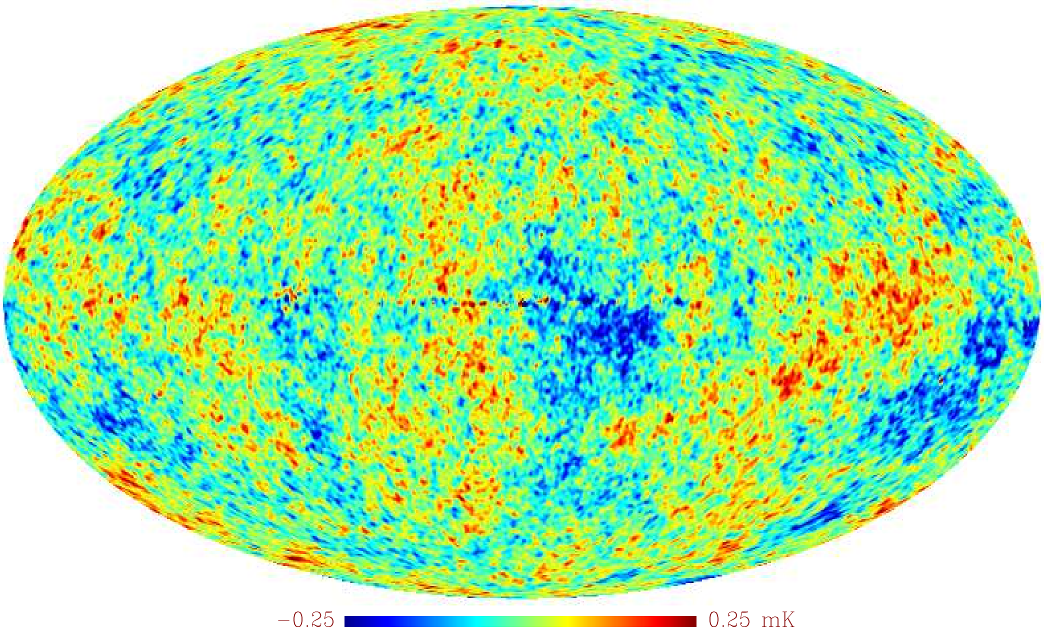

Figure 3: The observed tempertature of the cosmic microwave background measured in different directions on the sky. The map is in galactic coordinates. The points along the equator of this sphere are observaions made towards the galactic plane. Emission from the galaxy contaminates the observations of the background radiation in this direction. Signal processing techniques have been used to remove the galactic emission to 'see through' the galaxy in this picture.

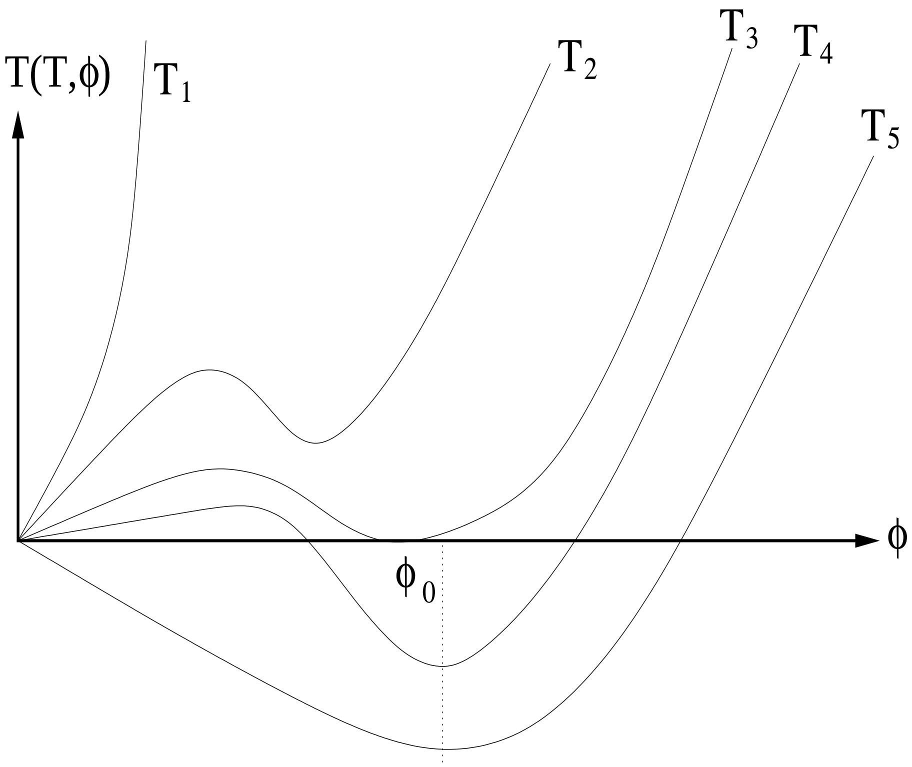

Figure 4: The potential of the quantum field ϕ as a function of ϕ and the temperature of the universe T. We see the shape of the potential for increasing temperature $T_1 > T_2 > T_3 > T_4 > T_5$. The minimum of the potential changes when the temperature falls below the criticual temperature T_c denoted T_3 in this figure.

been in contact gravitatonally, which they cannot have been. This is the horizon problem.

4 Inflation

In 1981 Alan Guth suggested a solution to these two problems using particle physics. He suggested that the universe started with a false vacuum state. In quantum physics, vacuum itself can have potential energy and an energy state. This happens if a so-called quantum field ϕ, a three-dimensional field, is present. In figure 5 we see an example of the possible potential energy of the vacuum. The x-axis shows the value of a quantum field ϕ. The y-axis shows the potential energy as a function of ϕ for a set of different temperatures T. The true shape of this potential as a function of the quantum field ϕ and temperature T is not known in detail, it is strongly dependent on unknown high energy particle physics.

At high temperatures T ($T = T_1$ in the figure), the field ϕ is zero which according to the figure is a stable minimum of the potential. At this temperature the vacuum was the true vacuum. But as the universe expanded, the temperature decreased and the potential $V(\phi, T)$ changed its shape. After a certian critical temperature ($T = T_3$), the potential had a new minimum which was lower than the minimum at $\phi = 0$. The vacuum was not anymore in the lowest possible energy state. The vacuum had acquired a constant energy density larger than the minimum energy in the universe defined by the minimum of the potential. We have previously seen what happens with space if the vacuum has a constant energy density. This is exactly what happens in the presence of dark energy. We deduced that the result is that space starts to expand exponentially $R(t) \propto e^{Ct}$. According to most theories this happened about $t \sim 10^{-36}$s after the Big Bang when the temperature of the universe was about $T \sim 10^{28}$K. The universe expanded expotentially until the quantum field had rolled down the potential to the new minimum at $\phi = \phi'$. The potential energy of the vacuum was released and matter and radiation was created from the vacuum. From this point, the classical hot Big Bang model with a dense plasma of particles and radiation discussed above starts.

It turns out that the period of expotential expansion of the universe a very short time after the Big Bang solves both the problems mentioned above. Inserting $R(t) \propto e^{Ct}$ where C is a constant in equation 2 above gives

$$(\Omega(t) - 1) \propto e^{-4ct}.$$

Clearly $\Omega \to 1$ rapidly, no matter what the value of Ω was just before inflation. The result of inflation is that the density is driven towards the critical density. This can be understood by the following analogy: If the radius of a curved sphere is increased sufficiently, a two-dimensional creature living on the surface of the sphere will experience the geometry of the sphere to be locally flat. This is exactly what we experience on Earth. The surface appears flat to us because of the huge size of the sphere. Inflation works in exactly the same manner: the scale factor $R(t)$ is increased so much that space locally looks flat to us and we measure $\Omega \approx 1$.

Inflation also solves the horizon problem. When we calculated the horizon at recombination, we assumed that the universe was matter dominated ($R(t) \propto t^{2/3}$) from the beginning. If inflation took place in the early universe, then $R(t) \propto e^{Ct}$ for a period and the horizon becomes much larger than if the universe had been only radiation/matter dominated. Two points in space could have been in causual contact before inflation. After inflation, the distance between these two points has increased immensly. Thus two points which are very far apart in space and which appears to be so far away that they have never been in causual contact, could have been in contact before inflation. Actually, calculations show that space could have expanded with a factor of up to e^{100} during the period of inflation which may have lasted about 10^{-34}s. Two points seperated by a distance similar to the atomic nucleus before inflation could be seperated by a distance larger than the current observable universe after inflation.

Inflation also solves another problem: where did the structures in the universe come from? The universe is clearly not completely homogenous at small scales, it consists of clusters of galaxies, galaxies and stars. These structures have formed by the gravitational collapse of overdense regions in the primordial plasma. But before the theory of inflation, there was no theory to explain how these overdensities were created. Inflation offers an easy explanation: quantum fluctuations. Tiny quantum fluctuations in the vacuum during inflation were inflated to much larger dimensions during inflation. Thus, at the end of inflation, the plasma contained some areas with higher and some areas with lower densities. The size of these overdensities were similar to the size of clusters of galaxies. By gravitational collapse these overdense regions became clusters of galaxies. Thus, by studying the largest structures in the universe today, we are studying quantum fluctuations at the beginning of the universe, fluctuations which are normally so small that they cannot be observed directly.

5 Growth of structure in the universe

We now explain the origin of structure in the universe with quantum fluctuations from inflation. But the density in these fluctuations is only slightly higher than the density in the sorrounding area. As time goes by, these density fluctuations, the areas in space with higher density than the mean density of the universe, will start collapsing due to their own gravitation. But the radiation pressure in the early radiation dominated phase of the universe was too high to enable these structures to grow to galaxies and clusters of galaxies today. The only way to solve this would be if there was a kind of matter present that does not react with normal matter. This matter would not feel the pressure from radiation and could thus grow much faster than normal matter during the radiation dominated phase. At the end of the radiation dominated phase, also the normal matter could start forming structure. The dark matter had already formed massive lumps with a large gravitational potential. When the radiation pressure was gone, the normal matter started falling into the gravitational potential from the invisible matter and clusters of galaxies could form. This invisible matter which is necessary for galaxies, stars and planets to form from the initial quantum fluctuations must have the same properties that the dark matter has. It has also been calculated that the amount of dark matter necessary to form clusters of galaxies equals the amount of dark matter observed in the universe today. This is another important confirmation of the existence of dark matter. The total energy density of the universe can be written as

$$\Omega_0 = \Omega_{b0} + \Omega_{DM0} + \Omega_{r0} + \Omega_{\Lambda 0} \approx 1,$$

where all densities are taken today at $t = t_0$. The density of baryons divided by the critical density is given by Ω_{b0}, the density of dark matter by Ω_{DM0}, the density of radiation Ω_{r0} and the energy density of dark energy $\Omega_{\Lambda 0}$. The current best estimates of these values from various observations are $\Omega_{b0} \approx 0.04$, $\Omega_{r0} \approx 10^{-4}$, $\Omega_{DM0} \approx 0.23$ and $\Omega_{\Lambda 0} \approx 0.73$.

6 Problems

Problem 1 We will go a little bit into details about the nucleosynthesis process and show some numbers that we used in the text. Before starting the exercise, read carefully through the chapter on nucleosynthesis.

1. Before the onset of nucleosynthesis, the process

$$^1_0\mathrm{n} \rightarrow ^1_1\mathrm{p} + ^0_{-1}\mathrm{e} + ^0_0\bar{\nu}_e. \tag{4}$$

which so far had been going in both direction, only went from left to right. That is, at some temperature T, neutrons were not anymore

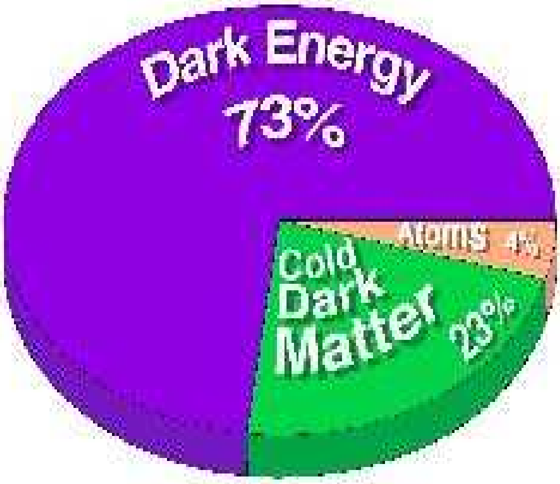

Figure 5: The 'ingredients' of the universe. Figure taken from NASA/WMAP homepage.

created by proton-electron collisions so the number density of neutrons could not increase. However, until nucleosynthesis started, the neutrons stilled decayed. Free neutrons decay with a half-life of 10.3 minutes. After nucleosynthesis, all neutrons were bound in atomic nuclei and the decay probability became practically zero. Thus, after nucleosythesis the number of neutrons in the universe was frozen until today. For this reason we can use nucelosynthesis to predict the number of neutrons present in the universe today. We will start by calculating the temperature T when neutrons could not anymore be produced from protons and electrons. Neglect the neutrino and use energy considerations to find the minimum temperature when the above process could still go from right to left (proton mass $= 1.6726 \times 10^{-27}$ kg, neutron mass $= 1.6749 \times 10^{-27}$ kg, electron mass $= 9.1 \times 10^{-31}$ kg). **hint:** assume ideal gas

2. In order for the above process (right to left) to run, there must be a high abundance of electrons available. At a temperature T, the process

$$ {}^{0}_{0}\gamma \rightleftharpoons {}^{0}_{-1}\,\mathrm{e} + {}^{0}_{1}\,\bar{\mathrm{e}}. $$

which so far had been going in both directions, could only go from right to left, not from left to right. The result is that electrons rapidly disappear from the plasma, no more electrons are created by this process. The mean energy per photon in a photon gas is kT. At what temperature did this process (left to right) cease?

3. In the previous two questions, you should have found two reasons why no more neutrons were created after a temperature of about $T = 10^{10}$ K. You should have found one process to cease before $T = 10^{10}$ K and one after. We will now assume that it caesed exactly at $T = 10^{10}$ K. In the text, we calculated that at this temperature there were 223 neutrons per 1000 protons (if you haven't already done it, now is the time to check that calculation). From now on, neutrons will disappear from the plasma and protons will be created. The process responsible for this is equation 4. This continues until the onset of nucleosynthesis at $T = 10^9$ K. If we can find the time it takes from $T = 10^{10}$ K to nucelosynthesis starts at $T = 10^9$ K we can calculate how many neutrons disappear and how many protons are produced. Use the fact that the density of a photon gas is given by $\rho_r = aT^4$ (a is the radiation constant) as well

as the equation for how the energy density of a photon gas decrease with the scale factor (which you deduced in the problems last week) to show that

$$T(t) \propto \frac{1}{R(t)},$$

where $T(t)$ is the temperature of the universe at time t and $R(t)$ is the scale factor.

4. Use the form of the scale factor in a flat radiation dominated universe to show that

$$\frac{T(t_1)}{T(t_2)} = \sqrt{\frac{t_2}{t_1}}$$

for the temperature and age at two times t_1 and t_2 in the radiation dominated epoch.

5. We want to find the age of the universe when $T = 10^{10}$ K and when $T = 10^9$ K. The difference between these to ages is the time it takes from the universe had a temperature of $T = 10^{10}$ K to $T = 10^9$ K. If we know either t_1 or t_2 we can use this equation to obtain the other. Before we can use this equation, we therefore need to find the time (say t_1) at which $T = 10^{10}$K. We can do this by using this equation with t_1 being the time when $T = 10^{10}$K and t_2 being some known time with known temperature. To find the age of the universe when $T = 10^{10}$ K we therefore need to compare with a period when we know both the age and the temperature of the universe. This period has to be in or close to the radiation dominated epoch so that the above equation is valid. We can use the period of matter-radiation equality. In the previous lecture, we calculated that the age of the universe at matter-radiation equality was about 60000 years. We do not know the temperature of the universe in this period but can easily find it by using another fact: we also calculated that the recombination period took place when the age of the universe was 360000 years and the temperature was 3000 K. But this was already in the matter dominated epoch. Show that for a matter dominated universe,

$$\frac{T(t_1)}{T(t_2)} = \left(\frac{t_2}{t_1}\right)^{2/3},$$

and use this to show that the temperature at matter-radiation equality was about 10000K.

6. Now you know the temperature and age of the universe at matter-radiation equality. Use these two facts to show that the age of the universe when $T = 10^{10}$K was $t \approx 2$ s.

7. Use the same equation to show that the universe was about 189 seconds old when nucleosynthesis started.

8. The equation for radiavtive decay is

$$\frac{n(t_1)}{n(t_2)} = e^{-\lambda(t_1 - t_2)},$$

where $n(t_1)$ is the number density of nuceli at time t_1 and $n(t_2)$ is the number density of nuceli at time t_2. Here t_2 is before t_1. The half life τ of the decay process is given by λ from the relation

$$\lambda = \frac{\ln 2}{\tau}.$$

For neutron decay, the half life is 10.3 minutes. Show that from the period when no more neutrons were created to nucelosynthesis started, a total of 181 remained of the original 223 neutrons. Show also that 1000 protons had grown to 1042 protons.

9. Finally show that the mass fraction X_{He} of helium after nucelosynthesis must be roughly 0.296.

Problem 2

In the previous lecture notes, we used that the luminosity distance can be written as

$$d_L = \frac{1}{H_0 q_0^2}[q_0 z + (q_0 - 1)(\sqrt{1 + 2zq_0} - 1)].$$

1. In a flat universe the value of q_0 is known. Use the previous expression to show that for the flat universe, the luminosity distance can be written as

$$d_L = \frac{2}{H_0}[1 + z - \sqrt{1 + z}].$$

2. Assume a flat pressureless universe (all objects that we observe emitted light in the era of the pressureless universe) and use the form of $R(t)$

in such a universe. Use equation (15) from the previous lecture for a light beam to show that the coordinate distance r that light emitted at time t and received at time t_0 has traveled is

$$r = \frac{3t_0}{R(t_0)}\left[1 - \left(\frac{t}{t_0}\right)^{1/3}\right]$$

for a flat pressureless universe.

3. Now use the expression relating $R(t_0)/R(t)$ to the redshift z and deduce the above expression for the luminosity distance in a flat pressureless universe.

4. Insert the expression for the luminosity distance (the general expression including q_0) into the relation between apparent and absolute magnitude and show that

$$m - M = 5\log(q_0 z + (q_0 - 1)(\sqrt{1 + 2zq_0} - 1)) - 10\log(q_0) - 5\log(H_0 \times 10\text{pc})$$

5. We will now make a plot with $m - M$ on the y-axis and redshift z on the x-axis. Include values of redshift up to $z = 2$. Assume the current value of the Hubble constant $H_0 = 71$ km/s/Mpc. Plot three models of the universe on the same plot, $\Omega_0 = 0.3$, $\Omega_0 = 1$ and $\Omega_0 = 1.5$. Constrain the range on the y-axis of the plot for values of $m - M$ between 37 and 46. Explain how you can use this plot, combined with observations of supernovae to find the geometry of the universe. Then, use the plot to answer the following two questions: (a) in order to find the geometry of the universe, would you observe nearby or distant supernovae ? (b) Roughly what minimum redshift should the supernovae that you observe have in order for you to easily find the geometry of the universe ?

6. In problem 3 of lecture notes 22, you used a simplified model of a supernova to find the absolute magnitude and thereby the distance of the supernova. Study this exercises once more since we will now use the same assumptions about the supernovae to find the geometry of the universe. In that problem, we observed the velocity v of the expanding shell and the temperature T of the shell a time Δt after the explotion. We will in the following use these observations, as well as the observed redshift z of supernovae to estimate the geometry of the universe.

7. Using this model of the expanding shell, show that you can write $m-M$ for the supernova as

$$m - M = m - 86.7 + 10\log_{10}(T) + 5\log_{10}(v)\log_{10}(\Delta t),$$

where m is apparent magnitude and M is absolute magnitude (not observed directly), the temperature T is measured in Kelvin, the shell velocity v is measured in m/s and the time delay Δt is measured in seconds.

8. You will now look at three files. Each file contains a list of 1000 observed (simulated) supernovae. Each file is for a different universe. The files can be found here:

$http://folk.uio.no/frodekh/AST1100/lecture25/supernovadata\_?.txt$

There are five columns in each file: first column is the redshift (the supernovae are ordered by increasing redshift), the second column is the apparent magnitude, the third column is the measured surface temperature of the expanding shell, the fourth column is the shell velocity in km/s and the last column is the time of observation measured in days after the supernova exploded. Load these data and construct an array which gives $m - M$ as a function of redshift. Make a plot with $m - M$ on the y-axis and redshift z on the x-axis for one of the universes. Use the same range on the y-axis for $m - M$ as indicated above. On top of this plot, plot the same three lines for the same three models that you plotted above. Can you use this plot to tell if the universe you are looking at is open, flat or closed? Repeat for the other two universes. Which of these three universes do you think is most similar to our universe?

9. You have now seen how you can use observations of supernovae to get an idea of the geometry of the universe. Now we will see how wrong you were and how easy it is to fool yourself if you don't make a thorough analysis of the data. We will now make a better analysis by the now well known least square fitting. The function on which we will make least square fitting is $(m-M)$. You have models of $m-M$ for different values of q_0 (and thereby Ω_0) and you have a set of observations of

$m - M$. So we will again try to find the model, that is, the value of q_0, which gives the smallest possible difference between the model and the data. Show that you can write the above expression for the model in terms of q_0 as

$$m-M = 5\log(q_0 z+(q_0-1)(\sqrt{1+2zq_0}-1))-5\log(q_0^2)-5\log(H_0\times 10\text{pc})$$

It is very important that you use exactly this expression in your computer code otherwise you may run into some numerical problems.

10. We can now construct the function to minimize, i.e. the sum of the square of the model minus the data, summed over all redshifts. This is given by

$$\begin{aligned}\Delta(q_0) &= \sum_{z=0}^{2}[(m-M)_{\text{data}} - (m-M)_{\text{model}}]^2 \\ &= \sum_{z=0}^{2}[(m-M)_{\text{data}} - (5\log(q_0 z + (q_0-1)(\sqrt{1+2zq_0}-1)) \\ &\quad -5\log(q_0^2) - 5\log(H_0 \times 10\text{pc}))]^2.\end{aligned}$$

You can now construct a code in exactly the same way as you did before with least square minimization: make an array for $\Delta(q_0)$. Then make a loop over a set of possible values of q_0 and calculate $\Delta(q_0)$ each time. You should choose the range of q_0 to go from $q_0 = -0.25$ to $q_0 = 2$. Use about 1000 different values for q_0 in this range. Then find the value of q_0 which minimizes $\Delta(q_0)$. Which Ω_0 does this value correspond to? You should get a surprise in one of these models. You will find in that model that the universe accelerates and you get a negative value for Ω_0. For that model, Ω_0 is clearly not negative, it is the relation between q_0 and Ω_0 which is wrong in this case. You will not learn the correct relation in this corse, but explain the physical reasons why we may get a negative q_0. What is happening?

11. Now we will return to the question above and see if you have changed your mind: which of these three sets of supernova observations do you think is most similar to observations in our universe?

12. We will finally look into a problem which we have with this way of measuring the geometry of the universe. Make the least square fitting

above for the model with negative q_0, but this time you only include the supernovae up to redshift $z = 0.2$ Which value of q_0 do you get? Why did you get a different value this time? Supernovae in the nearby universe $z < 0.2$ are much easier to find and observe than distant supernovae. We therefore have much more data for supernovae with low redshift. This can make measurements of q_0 uncertain. We have not included error bars in our analysis, but you have already got an idea of the size of the errors if we only include nearby supernovae.

13. We have used a very simple model of the supernova. A more sophisticated model is used in the real analysis. But these models are based on observations in the nearby universe. One of the big debates in cosmology novadays is whether we can trust that the same model for the supernova is true for the distant supernovae. Clearly if the model is wrong, also the q_0 obtained with least square fitting is wrong. Can you find a good reason why a model for supernovae which is found to be correct for nearby supernova is not necessarily correct for distant supernovae?