1 Introduction

In any flow, layers move at different velocities and a fluid’s viscosity arises from the shear stress between the layers that ultimately oppose any applied force. These shear stresses are communicated via collisions between the particles comprising the fluid, and the typical scale of viscosity is therefore of order the mean free collisional length $\lambda$. This scale is usually much smaller than the scales that can be represented on a numerical grid, $\lambda <\Delta x$ if one desires to model an astrophysical phenomenon of macroscopic size, and an understanding of this limitation is therefore vital to understanding the limitations of numerical modeling.

Figure 1: Sketch for illustration of discussion on viscosity: Variations in the velocity shear ($du_x/dy$) are communicated via random (thermal) motions in the fluid with mean free path $\lambda$ between collisions. This leads to a shear stress $\tau$ on area $A$ in the $x$ direction.

We follow Wikipedia (http://en.wikipedia.org/wiki/Viscosity) and consider the relation between the mean free path and the viscosity. For a Newtonian fluid, the shear stress $\tau$ on a unit area moving parallel to itself is (empirically) found to be proportional to the rate of change of velocity with distance perpendicular to the unit area parallel to the $x-z$ plane, moving along the $x$-axis:

$$\tau = \mu \frac{du_x}{dy}.$$
Interpreting shear stress as the time rate of change of momentum $p$ per unit area $A$ of an arbitrary surface gives

$$\tau = \dot{p}A = \dot{m} \langle u_x \rangle A$$

where $\dot{m}$ is the rate of fluid mass hitting the surface $A$ and $\langle u_x \rangle$ is the average velocity along $x$ of fluid molecules hitting the unit area; see figure 1 for a sketch of the geometry. We can write

$$\dot{m} = \rho \bar{u} A \quad \text{and} \quad \langle u_x \rangle = \frac{1}{2} \lambda \frac{du_x}{dy}$$

where $\rho$ is the density of the fluid and $\bar{u} = \sqrt{\langle u^2 \rangle}$ is the average particle speed. We are and assuming that particles hitting the unit area are equally distributed between distances 0 and $\lambda$ from $A$ and that their average velocities change linearly with distance (this will always be true with small enough $\lambda$). From this we get

$$\tau = \frac{1}{2} \rho \bar{u} \lambda \frac{du_x}{dy} \Rightarrow \nu = \frac{\mu}{\rho} = \frac{1}{2} \bar{u} \lambda. \quad (1)$$

The quantity $\mu$ is the *dynamic viscosity* while $\nu \equiv \mu/\rho$ is the *kinematic viscosity*. We will use later that $\bar{u} \sim c_s$, i.e. that the average speed of particles is the speed of sound in the medium.

### 1.1 Cauchy’s equation of motion

We need to see how the viscosity appears in the equation of motion for the fluid. A fluid element’s momentum is changed by both body forces and surface forces.

$$\frac{D}{Dt} \int_V \rho u dV = \int_V f dV + \int_S t dS$$

Using the equation of mass continuity we can rewrite this in component form as

$$\int_V \rho (Du^i / Dt) dV = \int_V f^i dV + \int_S t^i dS$$

or using the divergence theorem

$$\int_V \rho (Du^i / Dt) dV = \int_V f^i dV + \int_S T^{ij} n_j dS = \int_V (f^i + T^{ij}) dV$$

where we have written the surface force $t$ as $t_i = n^i T_{ji}$ with the unit vector $n$ representing the normal to the surface. Since the volume $V$ is arbitrary the integrals are only equal if their integrands are equal. Thus we get *Cauchy’s equation of motion*:

$$\rho (Du^i / Dt) = f^i + T^{ij}$$

### 2 The Stress Tensor in a Newtonian Fluid

We have shown the equation of motion to be dependent on the stress tensor $T_{ij}$. Following Mihalas & Mihalas *Foundations of Radiation Hydrodynamics* we can derive an explicit expression for the stress tensor in terms of the physical properties of the fluid and its state of motion. Mihalas & Mihalas state the following four physical considerations:

1. Internal frictional forces should only exist when one element of fluid moves relative to another; hence viscous terms must depend on the space derivates of the velocity field, $u_{i,j}$. This since the velocity at a small distance away from the point ($r$) can be written

$$u(r + dr) = u(r) + \frac{du}{dr} dr + \ldots$$

where $du/dr$ is shorthand for the dyadic product of the $\nabla$ operator and the velocity $u$. 

2. The stress tensor should reduce to its hydrostatic form when the fluid is at rest or translates uniformly. Therefore, we write
\[ T_{ij} = -p\delta_{ij} + \sigma_{ij} \]
where \( \sigma_{ij} \) is the **viscous stress tensor**, which accounts for the internal frictional forces in the flow.

3. For small velocity gradients we expect viscous forces, *i.e.* \( \sigma_{ij} \), to depend only linearly on space derivatives of the velocity. This is what is meant by a **Newtonian fluid**. A **Stokesian fluid** is one which depends quadratically on the velocity gradients.

4. Viscous forces will be zero within an element of fluid in rigid rotation because there is no slippage. On these grounds, there should be no contribution to \( \sigma_{ij} \) from the vorticity tensor \( \Omega_{ij} = (u_{i,j} - u_{j,i}) \).

The most general symmetrical tensor of rank two satisfying the above requirements is
\[ \sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \xi u^k_{k} \delta_{ij} \]
Assuming the fluid is isotropic, so that there are no preferred directions, implies that \( \xi \) and \( \mu \) must be scalars; \( \mu \) is also called the **coefficient of shear viscosity** and \( \xi \) is called the **dilational coefficient of viscosity**. This expression can also be written as
\[ \sigma_{ij} = \mu(u_{i,j} + u_{j,i} - \frac{4}{3} u_{k}^{k} \delta_{ij}) + \zeta u^k_{k} \delta_{ij} \]
where \( \zeta = \xi + \frac{2}{3} \mu \) is called the bulk viscosity and the expression in parenthesis is traceless. Fluids with no bulk viscosity, \( \zeta = 0 \), are so called **Maxwellian fluids**.

3 **Damping of (1D) acoustic waves by viscosity**

Let us now consider the effects of viscosity on a 1D planar flow. In this case we can write the momentum equation
\[ \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{4}{3}(\mu + \zeta) \frac{\partial u}{\partial x} \right] . \]

We can now define \( \nu' = (\mu + \frac{4}{3} \zeta) \) as the **effective viscosity coefficient**, which shows that the viscosity works just like a “pressure”, \( Q \), in the 1D equation with
\[ Q \equiv -\frac{4}{3} \nu' \left( \frac{\partial u}{\partial x} \right) . \] (2)

3.1 **Equation of state and speed of sound**

To make things simple let’s assume that we have a perfect gas **equation of state** (EOS) relating the pressure \( p \) and the internal energy \( e \)
\[ p = (\gamma - 1)e \]
where \( \gamma \) is the adiabatic index. Assume further that the flow is nearly adiabatic\(^1\). In this case we can write the linearized energy equation as
\[ \frac{1}{\gamma - 1} \frac{\partial p}{\partial t} + e_0 \frac{\partial u}{\partial x} = -p_0 \frac{\partial u}{\partial x} \]

\(^1\)this is of course not correct in our case since we have a non-zero viscosity and therefore non-adiabatic flow, but this does not matter too much for the arguments below.
where we assume that \( u_0 = 0 \) and therefore write \( u_1 \equiv u \). The linearized continuity equation

\[
\frac{\partial u}{\partial x} = -\frac{1}{\rho_0} \frac{\partial \rho_1}{\partial t}
\]

allows us to eliminate \( \partial u/\partial x \), while the EOS allows us to replace \( e \) with \( p \), giving

\[
\frac{1}{\gamma - 1} \frac{\partial p_1}{\partial t} - \frac{p_0}{\rho_0} \frac{1}{\gamma - 1} \frac{\partial \rho_1}{\partial t} = \frac{p_0}{\rho_0} \frac{\partial \rho_1}{\partial t} \quad \frac{\partial \rho_1}{\partial t} = \frac{\gamma p_0}{\rho_0} \frac{\partial \rho_1}{\partial t}
\]

which is equivalent to

\[
\Rightarrow \quad \frac{\partial}{\partial t}(p_1 + c_s^2 \rho_1) = 0
\]

which again implies that \( p_1 = c_s^2 \rho_1 \). We have defined the speed of sound \( c_s = \sqrt{\gamma p_0/\rho_0} \).

### 3.2 Solution of the linearized equations

Having derived a relation between the pressure and density, let us now assume a solution to the linearized equations of continuity and momentum

\[
\frac{\partial \rho_1}{\partial t} + \rho \frac{\partial u}{\partial x} = 0
\]

\[
\rho \frac{\partial u}{\partial t} + \frac{\partial p_1}{\partial x} + \frac{\partial}{\partial x} \left[ -\frac{4}{3} \mu' \left( \frac{\partial u}{\partial x} \right) \right] = 0
\]

of the form

\[
\rho_1 = Re^{i(\omega t - kx)} \quad p_1 = Pe^{i(\omega t - kx)} \quad u = Ue^{i(\omega t - kx)}
\]

Inserting this solution into the equations gives

\[
i\omega R - \rho_0 U i k = 0
\]
\[
i\omega \rho_0 U + R c_s^2 i k + \mu' \mu' k^2 U = 0
\]

If our solution is to be a valid, non-trivial solution we must require that the determinant of the coefficient matrix vanishes

\[
\begin{vmatrix}
    i\omega & -ik\rho_0 \\
    ikc_s^2 & i\omega \rho_0 + \mu' k^2 
\end{vmatrix} = 0,
\]

which leads to the dispersion relation

\[
-\omega^2 \rho_0 + i\omega k^2 \mu' + k^2 \rho_0 c_s^2 = 0
\]

\[
\Rightarrow \quad \omega^2 = c_s^2 k^2 + i\mu' k^2 \omega/\rho_0.
\]
3.3 Effect of viscosity on acoustic waves

Now assume that the effective viscosity coefficient $\mu'$ is small. This implies that we can expand the wavenumber to first order

$$k = k_0 + \delta k \approx (\omega/c_s) + \delta k.$$  

and then, writing

$$k^2 = (k_0 + \delta k)^2 = (\omega/c_s)^2 + 2\delta k \omega/c_s + \ldots$$

Now insert this into the dispersion relation 3 to get

$$\omega^2 = c_s^2 \left[(\omega/c_s)^2 + 2\delta k (\omega/c_s)\right] + i\mu' \left[(\omega/c_s)^2 + 2\delta k (\omega/c_s)\right] \omega/\rho_0.$$  

Retaining only terms of up to first order (i.e. discarding the term proportional to the product of the small numbers $\mu'$ and $\delta k$) we solve this expression for $\delta k$ to give

$$\delta k = -i\omega^2/2c_s^3\rho_0\mu'.$$

Now insert this expression into the solution for the variation in the density to get

$$\rho_1 = Re^{i(\omega t - kx)} = Re^{i(\omega t - k_0 x)} e^{-x/L}$$

where the damping length $L$ is given by

$$L = 2c_s^3\rho_0/\omega^2\mu' = 2c_s^3/\omega\nu = 2c_s/k_0^2\nu.$$  

This lets us write the damping length, using (for this one time) the symbol $\Lambda = 2\pi/k$ for the wavelength of the wave, as

$$L = 4/k_0^2\lambda = \Lambda \left(\frac{\Lambda}{\lambda}\right) \frac{1}{\pi^2} \quad \Rightarrow \quad \left(\frac{L}{\Lambda}\right) = \left(\frac{\Lambda}{\lambda}\right) \frac{1}{\pi^2} \quad \text{i.e. damping in of order} \quad \left(\frac{\Lambda}{\lambda}\right) \text{ wavelengths.}$$  

This result clearly shows that high frequency acoustic waves are more heavily damped than low frequency waves by viscosity.

4 Artificial viscosity

4.1 The Reynolds number

Consider the momentum equation in the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2}.$$  

When we non-dimensionalize this equation, dividing by $U^2$ and multiplying by $L$, the following dimensionless number turns up

$$\text{Re} \equiv UL/\nu.$$  

where $U$ and $L$ are typical velocities of the system under consideration. This is the Reynolds number, which indicates the relative importance of the convective and viscous terms in the momentum equation. Note that $L$ is any typical length scale in the system, and therefore that viscosity will dominate at some scale as $L$ is chosen small.

Assume the scale over which dissipation takes place is microscopic. The question can then be posed “What viscosity do we require in order to produce an observed flow?” Any chaotic dissipative motions can play the role of viscosity on some length or time scale.
4.2 The development of shocks

Riemann showed that it is possible to obtain a general (not only small amplitude) solution to the full non-linear equations for a travelling wave, in which all the physical properties and the fluid velocity are functions of a single argument $x \pm ut$, but where now the propagation speed $v$ at each point of the wave profile is a function of the wave velocity $u$ at that point of the disturbance. It is therefore possible to express any physical property as a function of another; e.g. $\rho = \rho(u)$. Assuming this, we can write the continuity equation as

\[
\left( \frac{\partial \rho}{\partial u} \right) \left( \frac{\partial u}{\partial t} \right) + \left[ u \frac{\partial \rho}{\partial u} + \rho \right] \frac{\partial u}{\partial x} = 0
\]

or

\[
\frac{\partial u}{\partial t} + \left[ u + \rho \left( \frac{\partial u}{\partial \rho} \right) \right] \frac{\partial u}{\partial x} = 0.
\] (4)

Likewise, we can write the momentum equation

\[
\frac{\partial u}{\partial t} + \left[ u + \frac{1}{\rho} \left( \frac{\partial p}{\partial \rho} \right) \right] \frac{\partial u}{\partial x} = 0.
\] (5)

But if this is correct, then we must have

\[
\frac{du}{d\rho} = \pm \left( \frac{dp}{d\rho} \right)^{1/2}/\rho = \pm c_s/\rho.
\] (6)

Hence, the general relation between the fluid velocity and the density or pressure in the wave is

\[
u = \pm \int_{\rho_0}^{\rho} \left( \frac{c_s}{\rho} \right) d\rho = \pm \int_{p_0}^{p} dp/\rho c_s,
\] (7)

where $\rho_0$ and $p_0$ are the ambient values in the undisturbed fluid. Using equation 6 in equation 4 or 5 we get

\[
\frac{\partial u}{\partial t} + (u \pm c_s) \left( \frac{\partial u}{\partial x} \right) = 0.
\] (8)

In the same way, by inverting $\rho(u)$, we can write the continuity equation

\[
\frac{\partial \rho}{\partial t} + \left[ \rho \left( \frac{\partial u}{\partial \rho} \right) + u \right] \frac{\partial \rho}{\partial x} = 0
\]

which can be rewritten using equation 6 to

\[
\frac{\partial \rho}{\partial t} + (u \pm c_s) \left( \frac{\partial \rho}{\partial x} \right) = 0.
\] (9)

Equations 8 and equation 9 have general solutions of the form

\[
u = F_1 [x - (u \pm c_s) t]
\]

and

\[
\rho = F_2 [x - (u \pm c_s) t]
\]

where $F_1$ and $F_2$ are arbitrary functions that fix the run of $u$ and $\rho$ at $t = 0$. These equations represent traveling waves known as *simple waves*. A particular value of, say, $\rho$ or $u$ propagates through the ambient medium with the phase speed

\[v_p(u) = u \pm c_s(u).
\]

We choose the positive (negative) sign for waves traveling in the positive (negative) $x$ direction.
4.2.1 A simple wave in a perfect gas

Consider a simple wave in a perfect gas. In this case

\[ c_s^2 \propto \frac{p}{\rho} \propto \rho^{\gamma-1} \Rightarrow (\gamma - 1)(\frac{d\rho}{\rho}) = 2(dc_s/c_s). \]

Using equation 7 the expressions above yields

\[ u = \pm 2(c_s - c_{s0})/(\gamma - 1), \]

or

\[ c_s = c_{s0} \pm \frac{1}{2}(\gamma - 1)u, \quad \text{which implies a phase speed} \]

\[ \Rightarrow v_p(u) = \frac{1}{2}(\gamma + 1)u \pm c_{s0}. \]

Using the polytropic gas laws we can also derive

\[ \rho = \rho_0 \left[ 1 \pm \frac{1}{2}(\gamma - 1)(u/c_{s0}) \right]^{2/((\gamma - 1))}, \]

\[ p = p_0 \left[ 1 \pm \frac{1}{2}(\gamma - 1)(u/c_{s0}) \right]^{2\gamma/((\gamma - 1))}. \]

Consider a finite pulse having an initial sinusoidal wavepacket moving in the positive x-direction. The more compressed part of the wave has a larger fluid velocity, is hotter, has a larger sound speed \( c_s(u) \), and moves with a larger phase velocity \( v_p(u) \). Thus, the crest of the wave gains on the pulse front and the wave steepens; note that there is nothing in the equation that prevents the wave from becoming infinitely steep.

This is a general behavior of all compressive acoustic waves. Our task as numerical modellers is therefore to come up with a method for controlling this behavior before the wave steepens so much that we no longer can represent it on our grid of necessarily limited spatial resolution.

4.2.2 Shock structure

Before looking into the techniques of using artificial viscosity to control the development of shocks in numerical simulations let’s consider the structure of hydrodynamic shocks beginning with the Rankine-Hugoniot relations.

We will be looking into steady shocks, i.e. we will transform our reference frame to one moving with the shock front itself, say at speed \( v_s \). Since the shock width (as we will prove here) is on the order of a few particle mean free paths \( \lambda \), it will be much smaller than the typical scale \( H \) of the system as long as the Knudsen number,

\[ Kn \equiv \lambda/H \]

is very small. If the lab frame velocity into which the shock propagates has speed \( v_1 \) and the material behind the shock has speed \( v_2 \) then in the lab frame upstream material enters the shock with speed \( u_1 = v_1 - v_s \) and downstream material moves with shock frame speed \( u_2 = v_2 - v_s \).

For a viscous 1D flow we have that across a shock we can write the conservation relations for a steady flow as

\[ \rho u = \rho_1 u_1 = \dot{m} \]

\[ \rho u^2 + p - \mu'(du/dx) = \rho_1 u_1^2 + p_1 \]

\[ \rho u \left( h + \frac{1}{2}u^2 \right) - \mu' u(du/dx) = \rho_1 u_1 \left( h_1 + \frac{1}{2}u_1^2 \right) \]
where $h \equiv (e + p)/\rho$. The entropy increase across the shock is given by

$$\rho u T \frac{ds}{dx} = \mu' \left( \frac{du}{dx} \right)^2. \quad (13)$$

If we evaluate the left hand sides of these equations far from the shock where $u \to u_2$, and where therefore $(du/dx) = 0$ we recover the ideal-fluid Rankine-Hugoniot jump relations.

Figure 2: Variation of $w = (u_1 - u)$ across a viscous shock, the thick line shows the estimate of the shock thickness $\delta \approx [w/(dw/dx)]_0$ derived in the text.

We can rewrite the jump relation for momentum, equation 11, as

$$u \left\{ \frac{1}{2} \dot{m} u + [\gamma p/(\gamma - 1)] \right\} - \mu' u (du/dx) = \dot{m} \left\{ \frac{1}{2} u_1^2 + [\gamma p/(\gamma - 1) \rho_1] \right\}$$

Now multiply equation 11 by $\gamma u/(\gamma - 1)$ and subtract the result from the expression above to find

$$-\nu u (du/dx) = c_{s1}^2 (u - u_1) + u_1^2 \left[ \gamma u - \frac{1}{2} (\gamma - 1) u_1 \right] - \frac{1}{2} (\gamma + 1) u_1 u^2,$$

where we remember that $\nu = \mu'/\rho$ is the effective kinematic viscosity. Now define $w \equiv u_1 - u$ and rewrite the expression above as

$$\nu (dw/dx) = w \left[ u_1^2 - c_{s1}^2 - \frac{1}{2} (\gamma + 1) u_1 w \right] / (u_1 - w). \quad (14)$$

The velocity drop $w$ varies from $w = 0$ far upstream to

$$w_{\text{max}} = u_1 - u_2 = 2 \left( u_1^2 - c_{s1}^2 \right) / (\gamma + 1) u_1 \quad (15)$$
far downstream. We have used the Prandtl relation: \( u_1 u_2 = u_2' \), with the critical velocity \( u_c \) being the velocity where the flow speed equals the local sound speed \( c_s \) in obtaining the last relation. These expressions also show that \( (dw/dx) \geq 0 \) with \( (dw/dx) = 0 \) at \( w = 0 \) and \( w = w_{\text{max}} \); \( w(x) \) is a monotone increasing and therefore \( u(x) \) is monotone decreasing. Furthermore, \( w \) has an inflection point because

\[
\nu (d^2w/dx^2) = u_0 \left[ u_1^2 - c_{s_1}^2 - (\gamma + 1) w \left( u_1 - \frac{1}{2} w \right) \right] / (u_1 - w)^2,
\]
i.e. that \( (d^2w/dx^2) > 0 \) at \( w = 0 \), \( (d^2w/dx^2) < 0 \) at \( w = w_{\text{max}} \), and \( (d^2w/dx^2) = 0 \) at \( w = u_1 - \sqrt{u_1 u_2} \).

An estimate of the shock width \( \delta \) can now be made by setting

\[
\delta \approx \left[ w / (dw/dx) \right]_{x_0},
\]
where \( x_0 \) is the point at which \( w = \frac{1}{2} w_{\text{max}} \). Substituting the expression for \( w_{\text{max}} \), equation 15, into equation 14 this gives

\[
(dw/dx)_{x_0} = \frac{1}{2} (u_1^2 - c_{s_1}^2)^2 / \nu (\gamma u_1^2 + c_{s_1}^2),
\]
and we estimate a shock width

\[
\delta = 2\nu (\gamma u_1^2 + c_{s_1}^2) / [(\gamma + 1) u_1 (u_1^2 - c_{s_1}^2)].
\]

An example of this estimate compared with the shock structure given by equation 14 is shown in figure 2.

We already know that \( \nu \sim c_{s_1} \lambda \). In the weak shock limit \( u_1 \gg c_{s_1} \) we can derive the shock width

\[
\delta \approx \frac{2\nu}{c_{s_1}^2 (M_1^2 - 1)} = \frac{4\nu \gamma}{(\gamma + 1) c_{s_1} P} \sim \left( \frac{2\gamma}{\gamma + 1} \right) \left( \lambda / P \right),
\]
where \( M_1 \) is the Mach number of the flow and \( P \) is the fractional pressure jump across the shock.

For a strong shock \( (M_1 \gg 1) \) and we get a shock width

\[
\delta \approx \left( \frac{2\gamma}{\gamma + 1} \right) \left( \lambda / M_1 \right),
\]
which is wrong, since it predicts that \( \delta \) becomes smaller than \( \lambda \) when \( M_1 \gg 1 \), which is incompatible with the fluid description, and comes from assuming \( \nu \sim c_{s_1} \lambda \). Instead, we should reason that most of the viscous dissipation occurs in the hotter material at the back of the shock, which gives a better estimate of \( \nu \sim c_{s_2} \lambda \sim M_1 c_{s_1} \lambda \). Thus,

\[
\delta \sim C \lambda
\]
where \( C \) is a number of order unity for a strong shock.

Entropy generation in shocks. Equation 13 shows that for a purely viscous shock we have

\[
\rho u T \left( \frac{ds}{dx} \right) = \dot{m} T \left( \frac{ds}{dx} \right) = \mu' \left( \frac{du}{dx} \right)^2.
\]
Thus, the entropy increases monotonically across the shock. This equation can be used to estimate the entropy jump across the shock by replacing the derivatives with finite differences:

\[
\dot{m} T_1 (\Delta s / \Delta x) \approx \mu' (u_2 - u_1)^2 / \Delta x^2
\]
The jump in velocity is \( \Delta u = (u_2 - u_1) = \Delta p / \dot{m} \approx \Delta p / \rho c_{s_1} \), for a weak shock where \( u_1 \approx c_{s_1} \). If we now set \( \Delta x \approx \delta \) we find, using equation 16, that

\[
\dot{m} T_1 \Delta s \approx \mu' \left( \frac{\Delta p}{\dot{m}} \right)^2 \left( \frac{\gamma + 1}{\rho} \right) c_{s_1} P / 4\gamma \nu,
\]
or

\[ T_1 \Delta s \approx \left( \frac{p_1}{\rho} \right)^2 \gamma + 1 \, \frac{p^3}{4 \gamma}. \]

Now using \( e = c_v \rho T, \, p = (\gamma - 1)e \) and \( c_s = \gamma p / \rho \) we can write the entropy increase across the shock as

\[ \Delta s = c_v \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{p^3}{4 \gamma^2}. \]

Note that the increase in entropy in the shock is independent of the value of the viscosity; in fact it is the same result as one would get when considering the change in entropy without including the viscosity at all in the analysis (see e.g. Mihalas & Mihalas Foundations of Radiation Hydrodynamics upon which much of the previous discussion is based). This perhaps surprising result is used when constructing schemes for handling shocks in numerical methods.

4.3 von Neumann-Richtmyer artificial viscosity

In order to achieve smoothness in our numerical solutions we will introduce a artificial viscosity as our dissipation mechanism. The 1D momentum and energy equations can be written as

\[ \rho \left( \frac{Du}{Dt} \right) = f - \left[ \frac{\partial (p + Q)}{\partial x} \right] \]

and

\[ \left( \frac{De}{Dt} \right) + (p + Q) \left[ \frac{D1/\rho}{Dt} \right] = q \]

where \( q \) is the energy input per unit mass from “external” sources (e.g. from radiation, conduction or joule heating) and \( Q \) is the equivalent viscous pressure given in equation 2.

One possibility would be to use a large value of \( \mu' \), chosen such that the artificial “mean free path” \( \lambda \) would be of order the grid spacing \( \Delta x \). This is not a good procedure however, since the shock thickness, for a given \( \lambda \), is inversely proportional to its strength, and we would obtain sharp strong shocks, but weak shocks would be spread over many grid points. In addition, such a large viscosity would spuriously reduce the Reynolds number of the flow in regions devoid of shocks and would therefore degrade the quality of the overall solution. Von Neumann and Richtmyer came up with a non-linear artificial viscosity that is large in shocks by very small otherwise. In particular they use a \( Q \) that is quadratic in the rate of shear:

\[ Q = \begin{cases} \frac{4}{3} \rho_l^2 \left( \frac{\partial u}{\partial x} \right)^2 & \text{for } \left( \frac{\partial u}{\partial x} \right) < 0, \\ 0 & \text{for } \left( \frac{\partial u}{\partial x} \right) \geq 0. \end{cases} \]  \hspace{1cm} (17)

Since \( \mu \) had dimensions \([g \text{ cm}^{-3}][\text{cm}^2\text{s}^{-1}]\), \( l \) must have dimensions of length. Typically \( l \) is chosen to be some small multiple of the grid spacing \( \Delta x \).

The artificial viscosity as given by this method only comes into action when the gas is compressed, and is zero or very small in regions away from shocks. A large body of computational work has demonstrated that the von Neumann-Richtmyer method gives good results as long as the resulting shock thickness, say 3 to 4 \( \Delta x \), is not too coarse to permit an accurate representation of other physical processes of interest (e.g. radiation transport).