UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**27. November 2017**
Exam hours:	**14.30-18.30, 4 hours**
This examination paper consists of 3 pages	
Permitted materials:	**Approved electronic calculator.**
	Angell and Lian: Størrelser og enheter i fysikken
	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1
Two interacting Two Level Systems

We have two interacting Two Level Systems, which we call systems A and B, with their corresponding sets of Pauli matrices σ_i^A and σ_i^B. The Hamiltonian is the following:

$$H = \frac{1}{2}\hbar g \sigma_z^A \otimes \sigma_z^B$$

where g is the interaction strength. Here we use a representation where for each system $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

a) Find the time evolution operator $U(t) = e^{-\frac{i}{\hbar}Ht}$ in the form of a 4×4 matrix.

b) Assume that at time $t = 0$ the two systems are in a product state

$$|\psi(0)\rangle = |\psi^A(0)\rangle \otimes |\psi^B(0)\rangle$$

with

$$|\psi^A(0)\rangle = a|0\rangle + b|1\rangle \qquad \text{and} \qquad |\psi^B(0)\rangle = c|0\rangle + d|1\rangle.$$

with $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$. Find the reduced density matrices for systems A and B as functions of time.

c) We define the Bloch vectors of A and B as $\mathbf{m}$ and $\mathbf{n}$, respectively, so that

$$\rho^A = \frac{1}{2}\left(\mathbb{1} + \mathbf{m}\cdot\sigma^A\right) \qquad \text{and} \qquad \rho^B = \frac{1}{2}\left(\mathbb{1} + \mathbf{n}\cdot\sigma^B\right)$$

Consider now the special case $a = b = \frac{1}{\sqrt{2}}$. Find the Bloch vector $\mathbf{m}$ for system A and show that as a function of time it is describing an ellipse in the xy-plane.

d) For given initial values c and d for system B and still $a = b = \frac{1}{\sqrt{2}}$, find the maximal value of the entanglement entropy, and show that it depends only in the component n_z of the Bloch vector $\mathbf{n}$ for system B.

PROBLEM 2

Squeezed states of the harmonic oscillator

We have in the lectures studied coherent states of the harmonic oscillator as examples of minimal uncertainty states. Here we will consider a related class of minimal uncertainty states called squeezed states. We define the squeeze operator

$$S(\zeta) = e^{-\frac{1}{2}(\zeta\hat{a}^2 - \zeta^*\hat{a}^{\dagger 2})}$$

where ζ is a complex number and $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators of the harmonic oscillator. The squeezed vacuum state is defined as

$$|sq_\zeta\rangle = S(\zeta)|0\rangle$$

a) Show that the action of the squeeze operator on $\hat{a}$ and $\hat{a}^\dagger$ is given by

$$\begin{aligned} S^\dagger(\zeta)\hat{a}S(\zeta) &= \cosh r\hat{a} + e^{-i\phi}\sinh r\hat{a}^\dagger \\ S^\dagger(\zeta)\hat{a}^\dagger S(\zeta) &= \cosh r\hat{a}^\dagger + e^{i\phi}\sinh r\hat{a} \end{aligned}$$

where $\zeta = re^{i\phi}$.

b) In the state $|sq_\zeta\rangle$, find the variance of the position and momentum operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right).$$

That is, calculate

$$\begin{aligned} \Delta x^2 &= \langle sq_\zeta|\hat{x}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{x}|sq_\zeta\rangle^2 \\ \Delta p^2 &= \langle sq_\zeta|\hat{p}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{p}|sq_\zeta\rangle^2 \end{aligned}$$

c) The Heisenberg uncertainty relation tells us that $\Delta x \Delta p \geq \frac{\hbar}{2}$ with equality only for minimal uncertainty states. Calculate the product $\Delta x \Delta p$ for the states $|sq_\zeta\rangle$ and show that for certain ϕ they are minimal uncertainty states.

d) For those ϕ which gives minimal uncertainty, compare Δx and Δp with the corresponding values in vacuum and describe what happens to the uncertainties.

e) For a general value of ϕ the state $|sq_\zeta\rangle$ is not of minimal uncertainty with respect to the operators $\hat{x}$ and $\hat{p}$. However, for any ϕ we can find transformed operators $\hat{x}_\phi$ and $\hat{p}_\phi$ satisfying the usual commutator relation $[\hat{x}_\phi, \hat{p}_\phi] = i\hbar$ and where $\Delta x_\phi \Delta p_\phi = \frac{\hbar}{2}$. Here Δx_ϕ and Δp_ϕ are defined by the same equations as Δx and Δp with $\hat{x}$ and $\hat{p}$ replaced by $\hat{x}_\phi$ and $\hat{p}_\phi$. Determine $\hat{x}_\phi$ and $\hat{p}_\phi$ expressed in terms of ϕ, $\hat{x}$ and $\hat{p}$.

We remind you of the general relation

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2}[B, [B, A]] + \cdots$$