

Fys 4110 exam 2017 Solutions.

Problem 1.

$$g) H = \frac{1}{2} g \sigma_2^A \otimes \sigma_2^B = \frac{1}{2} g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = e^{-\frac{i}{\hbar} H t} = \begin{pmatrix} z^* & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^* \end{pmatrix} \quad \text{where } z = e^{\frac{i \pi t}{2}}$$

$$|z| = 1$$

b) Alternative 1 (Brute force)

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ b \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

$$|\psi(t)\rangle = U |\psi(0)\rangle = \begin{pmatrix} z^*ac \\ z^*ad \\ z^*bc \\ z^*bd \end{pmatrix}$$

$$\beta = |\psi\rangle \langle \psi| = \begin{pmatrix} z^*ac \\ z^*ad \\ z^*bc \\ z^*bd \end{pmatrix} (z^*a^*c^*, z^*a^*d^*, z^*b^*c^*, z^*b^*d^*)$$

$$= \begin{pmatrix} |ac|^2 & z^{*2}|a|^2cd^* & z^{*2}ab^*c^* & ab^*cd^* \\ z^2|a|^2c^*d & |ad|^2 & ab^*c^*d & z^*ab^*ld^* \\ z^2a^*b^*c^*l^2 & a^*bcd^* & |bc|^2 & z^2|b|^2cd^* \\ a^*bc^*d & z^{*2}a^*b^*d^* & z^{*2}|b|^2c^*d & |bd|^2 \end{pmatrix}$$

$$S_A = \text{Tr}_B \beta = \begin{pmatrix} |a|^2 & ab^*(z^{*2}|c|^2 + z^2|d|^2) \\ a^*b(z^*|c|^2 + z^{*2}|d|^2) & |b|^2 \end{pmatrix}$$

$$S_B = \text{Tr}_A \beta = \begin{pmatrix} |c|^2 & cd^*(z^{*2}|a|^2 + z^2|b|^2) \\ c^*d(z^2|a|^2 + z^{*2}|b|^2) & |d|^2 \end{pmatrix}$$

Alternative 2 (More sophisticated, but not really simpler...)

With $z = x + iy$ we find

$$U = x \mathbb{1}^A \otimes \mathbb{1}^B - iy \sigma_2^A \otimes \sigma_2^B$$

$$\begin{aligned} f(t) &= |\mathcal{N}(t) \geq \mathcal{N}(0)| = U \underbrace{|\mathcal{N}(0) \geq \mathcal{N}(0)|}_{U^\dagger} U^\dagger \\ &\quad f(0) = f^A(0) \otimes f^B(0) \end{aligned}$$

$$\text{Let } f^A(0) = \frac{1}{2} (\mathbb{1} + \vec{m} \cdot \vec{\sigma}) \quad f^B(0) = \frac{1}{2} (\mathbb{1} + \vec{n} \cdot \vec{\sigma})$$

$$\begin{aligned} f(t) &= (x \mathbb{1}^A \otimes \mathbb{1}^B - iy \sigma_2^A \otimes \sigma_2^B) f^A(0) \otimes f^B(0) (x \mathbb{1}^A \otimes \mathbb{1}^B + iy \sigma_2^A \otimes \sigma_2^B) \\ &= x^2 f^A(0) \otimes f^B(0) + y^2 \sigma_2^A \otimes \sigma_2^B f^A(0) \otimes f^B(0) \sigma_2^A \otimes \sigma_2^B \\ &\quad + ixy [f^A(0) \otimes f^B(0) \sigma_2^A \otimes \sigma_2^B - \sigma_2^A \otimes \sigma_2^B f^A(0) \otimes f^B(0)] \\ &= x^2 f^A(0) \otimes f^B(0) + y^2 \sigma_2^A f^A(0) \sigma_2^A \otimes \sigma_2^B f^B(0) \sigma_2^B \\ &\quad + ixy [f^A(0) \sigma_2^A \otimes f^B(0) \sigma_2^B - \sigma_2^A f^A(0) \otimes \sigma_2^B f^B(0)] \end{aligned}$$

We have

$$\text{Tr } f^A(0) = 1$$

$$\text{Tr } \sigma_2^A f^A(0) \sigma_2^A = \frac{1}{2} \text{Tr } \sigma_2^A (\mathbb{1} + \vec{m} \cdot \vec{\sigma}) \sigma_2^A = 1$$

$$\text{Tr } f^A(0) \sigma_2^A = \frac{1}{2} \text{Tr} (\mathbb{1}_2^A + \vec{m} \cdot \vec{\sigma} \sigma_2^A) = m_2 = \text{Tr } \sigma_2^A f^A(0)$$

and similar for system B

$$\Rightarrow S^A(t) = \text{Tr}_B S = x^2 g^A(0) + y^2 \sigma_2^A g^A(0) \bar{\sigma}_2^A + ixy [g_A(0), \sigma_2^A]$$

$$= \frac{1}{2} [1 + (m_x \cos gt - m_y u_z \sin gt) \sigma_x^A \\ + (m_y \cos gt + m_x u_z \sin gt) \sigma_y^A + m_z \sigma_z^A]$$

$$S^B(t) = \frac{1}{2} [1 + (n_x \cos gt - n_y u_z \sin gt) \sigma_x^B \\ + (n_y \cos gt + n_x u_z \sin gt) \sigma_y^B + u_z \sigma_z^B]$$

9) Alternative 1

Using $z^2 = e^{igt} = \cos gt + i \sin gt$ and $a = b = \frac{1}{\sqrt{2}}$:

$$g^A = \frac{1}{2} \begin{pmatrix} 1 & \cos gt (\underbrace{|c|^2 + |d|^2}_1) - i \sin gt (\underbrace{|c|^2 - |d|^2}_{m_z}) \\ \text{c.c.} & 1 \end{pmatrix}$$

$$= \frac{1}{2} (1 + \cos gt \sigma_x^A + u_z \sin gt \sigma_y^A)$$

$$\Rightarrow m_x(t) = \cos gt \quad m_y(t) = u_z \sin gt \quad m_z(t) = 0$$

$$m_x(t)^2 + \left(\frac{m_y(t)}{u_z}\right)^2 = 1 \quad \Rightarrow \text{ellipse.}$$

Alternative 2.

$$g^A(0) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} (a^* b^*) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (11) = \frac{1}{2} (11) = \frac{1}{2} (1 + \sigma_x)$$

$$\Rightarrow m_x = 1, \quad m_y = m_z = 0$$

$$S^A(t) = \frac{1}{2} (1 + \cos gt \sigma_x^A + u_z \sin gt \sigma_y^A)$$

d) Maximal entanglement when the Bloch-vector is shortest $\Rightarrow g\hat{t} = \frac{\pi}{2}$ $\cos g\hat{t} = 0$ $\sin g\hat{t} = 1$.

$$\mathcal{G}^A(t) = \frac{1}{2} (I + u_2 \sigma_y^A) = \frac{1}{2} \begin{pmatrix} 1 & -iu_2 \\ iu_2 & 1 \end{pmatrix}$$

Eigenvalues: $(\frac{1}{2} - \lambda)^2 - (\frac{u_2}{2})^2 = 0 \Rightarrow \lambda_{\pm} = \frac{1}{2}(1 \pm u_2)$

$$S_{\max}^z = -\frac{1+u_2}{2} \ln \frac{1+u_2}{2} - \frac{(-u_2)}{2} \ln \frac{1-u_2}{2}$$

$$= \ln 2 - \frac{1}{2} \left[(1+u_2) \ln(1+u_2) + (-u_2) \ln(1-u_2) \right] = \begin{cases} 0 & u_2 = \pm 1 \\ \ln 2 & u_2 = 0 \end{cases}$$

Problem 2

g) $S(\beta) = e^{-\frac{1}{2}(\beta a^2 - \beta^* a^{*2})} \quad B = \frac{1}{2}(\beta a^2 - \beta^* a^{*2})$
 $B^+ = -B$

$$S^+ a S = e^B a e^{-B} = a + [B, a] + \frac{1}{2} [B, [B, a]] + \dots$$

$$[B, a] = -\frac{1}{2} \beta^* [a^2, a] = -\frac{1}{2} \beta^* (a^* [a^+, a] + [a^+, a] a^+) = \beta^* a^+$$

$$[B, a^+] = \frac{1}{2} \beta [a^2, a^+] = \frac{1}{2} \beta (a [a, a^+] + [a, a^+] a) = \beta a$$

$$S^+ a S = a + \beta^* a^+ + \frac{1}{2} \beta^* \beta a + \frac{1}{3!} \beta^{*2} \beta^2 a^+ + \frac{1}{4!} \beta^* \beta^2 a + \dots$$

$$= [1 + \frac{1}{2!} |\beta|^2 + \frac{1}{4!} |\beta|^4 + \dots] a + [\beta^* + \frac{1}{3!} \beta^{*2} \beta + \frac{1}{5!} \beta^* \beta^2 + \dots] a^+$$

$$= [1 + \frac{1}{2!} r^2 + \frac{1}{4!} r^4 + \dots] a + e^{-i\phi} [r + \frac{1}{3!} r^3 + \frac{1}{5!} r^5 + \dots] a^+$$

$$= \cosh r \cdot a + e^{-i\phi} \sinh r a^+$$

$$S a S = \cosh r \cdot a^+ + e^{i\phi} \sinh r a$$

(5)

$$b) \langle s_{q_3} | x | s_{q_3} \rangle = \langle 0 | S^+ \times S | 0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | S^+ (a^\dagger + a) S | 0 \rangle \\ = \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | (\cosh r + e^{-i\phi} \sinh r) a^\dagger + (\cosh r + e^{i\phi} \sinh r) a | 0 \rangle = 0$$

$$\langle s_{q_3} | p | s_{q_3} \rangle = \langle 0 | S^+ p S | 0 \rangle = i \sqrt{\frac{\hbar m\omega}{2}} \langle 0 | S^+ (a^\dagger - a) S | 0 \rangle \\ = i \sqrt{\frac{\hbar m\omega}{2}} \langle 0 | (\cosh r - e^{-i\phi} \sinh r) a^\dagger - (\cosh r - e^{i\phi} \sinh r) a | 0 \rangle = 0$$

$$\Delta x^2 = \langle s_{q_3} | x^2 | s_{q_3} \rangle = \langle 0 | S^+ \times S S^+ \times S | 0 \rangle \\ = \frac{\hbar}{2m\omega} (\cosh r + e^{i\phi} \sinh r)(\cosh r + e^{-i\phi} \sinh r) \\ = \frac{\hbar}{2m\omega} \left[\frac{\cosh^2 r + \sinh^2 r}{\cosh 2r} + \frac{\cosh r \sinh r}{\frac{1}{2} \sinh 2r} \underbrace{(e^{i\phi} + e^{-i\phi})}_{2 \cos \phi} \right] \\ = \frac{\hbar}{2m\omega} (\cosh 2r + \sinh 2r \cos \phi)$$

$$\Delta p^2 = \langle s_{q_3} | p^2 | s_{q_3} \rangle = \langle 0 | S^+ p S S^+ p S | 0 \rangle \\ = \frac{\hbar m\omega}{2} (\cosh r - e^{i\phi} \sinh r)(\cosh r - e^{-i\phi} \sinh r) \\ = \frac{\hbar m\omega}{2} \left[\cosh^2 r + \sinh^2 r - \cosh r \sinh r \underbrace{(e^{i\phi} + e^{-i\phi})}_{2 \cos \phi} \right] \\ = \frac{\hbar m\omega}{2} (\cosh 2r - \sinh 2r \cos \phi)$$

(6)

$$\text{6) } \Delta x \Delta p = \frac{\hbar}{2} \sqrt{\cosh^2 r - \sinh^2 r \cos^2 \phi}$$

$$= \frac{\hbar}{2} \sqrt{\cosh^2 r - \sinh^2 r (1 - \sin^2 \phi)}$$

$$= \frac{\hbar}{2} \sqrt{1 + \sinh^2 r \sin^2 \phi}$$

Minimal uncertainty: $\Delta x \Delta p = \frac{\hbar}{2}$

$$\Rightarrow \sin \phi = 0 \quad \Leftrightarrow \quad \phi = n\pi$$

d) For $\phi = n\pi$:

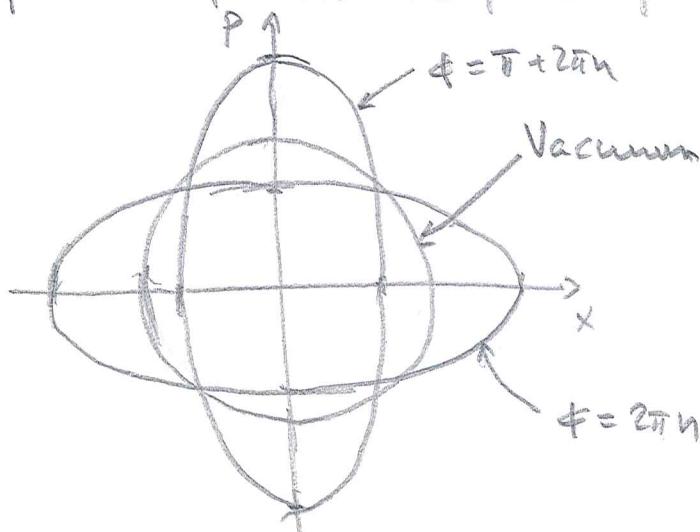
$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\cosh 2r + (-1)^n \sinh 2r} = \sqrt{\frac{\hbar}{2m\omega}} e^{(-1)^n r}$$

$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}} \sqrt{\cosh 2r - (-1)^n \sinh 2r} = \sqrt{\frac{\hbar m\omega}{2}} e^{-(-1)^n r}$$

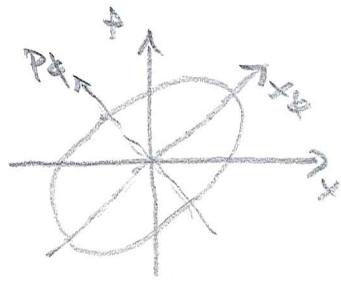
For n even Δx increases by a factor e^r
 Δp decreases by a factor e^r

For n odd Δx decreases and Δp increases.

Spread of wavefunction in phase space (Wigner function)



9) We guess that for other ϕ the wavefunction is spreaded in a direction not parallel to the axes. Thus we want to define "rotated" operators X_ϕ and p_ϕ . For this to be meaningful we introduce coordinates with same dimensions



(7)

$$\tilde{z} = x \sqrt{m\omega} = \sqrt{\frac{\hbar}{2}} (a^\dagger + a)$$

$$\tilde{\pi} = \frac{p}{\sqrt{m\omega}} = i\sqrt{\frac{\hbar}{2}} (a^\dagger - a)$$

Coordinates rotated by angle α :

$$\tilde{z}_\alpha = \cos\alpha \tilde{z} - \sin\alpha \tilde{\pi}$$

$$\tilde{\pi}_\alpha = \sin\alpha \tilde{z} + \cos\alpha \tilde{\pi}$$

$$\text{From b): } \langle S_{f_3} | \tilde{z}^2 | S_{f_3} \rangle = \frac{\hbar}{2} [\cosh 2r + \sinh 2r \cos \phi]$$

$$\langle S_{f_3} | \tilde{\pi}^2 | S_{f_3} \rangle = \frac{\hbar}{2} [\cosh 2r - \sinh 2r \cos \phi]$$

$$\langle S_{f_3} | \tilde{z}_\alpha | S_{f_3} \rangle = \langle S_{f_3} | \tilde{\pi}_\alpha | S_{f_3} \rangle = 0$$

$$\langle S_{f_3} | \tilde{z}_\alpha^2 | S_{f_3} \rangle = \langle S_{f_3} | \cos^2 \alpha \tilde{z}^2 - \cos \alpha \sin \alpha (\tilde{z}\tilde{\pi} + \tilde{\pi}\tilde{z}) + \sin^2 \alpha \tilde{\pi}^2 | S_{f_3} \rangle$$

We need to find

$$\langle S_{f_3} | \tilde{z}\tilde{\pi} | S_{f_3} \rangle = \langle 0 | S^\dagger \tilde{z} S S^\dagger \tilde{\pi} S | 0 \rangle$$

$$= i \frac{\hbar}{2} (\cosh r + e^{i\phi} \sinh r)(\cosh r - e^{-i\phi} \sinh r)$$

$$= i \frac{\hbar}{2} \underbrace{[\cosh^2 r - \sinh^2 r]}_1 + \underbrace{\cosh r \sinh r}_{\frac{i}{2} \sinh 2r} \underbrace{(e^{i\phi} - e^{-i\phi})}_{2i \sin \phi}$$

$$= \frac{\hbar}{2} (i - \sinh 2r \sin \phi) = \langle S_{f_3} | \tilde{\pi} \tilde{z} | S_{f_3} \rangle^*$$

$$\Rightarrow \Delta \tilde{z}_x^2 = \frac{\hbar}{2} \left[\cos^2 \alpha (\cosh 2r + \sinh 2r \cos \phi) + \sinh^2 \alpha (\cosh 2r - \sinh 2r \cos \phi) \right. \\ \left. + \cos \alpha \sin \alpha \sinh 2r \sin \phi \right] \\ = \frac{\hbar}{2} [\cosh 2r + \sinh 2r \cos(2\alpha - \phi)] \quad (8)$$

Similarly we find

$$\Delta \tilde{h}_\alpha^2 = \frac{\hbar}{2} [\cosh 2r - \sinh 2r \cos(2\alpha - \phi)]$$

We reproduce the minimal uncertainty expressions from d) if we choose $2\alpha - \phi = 0 \Rightarrow \alpha = \phi/2$

We should check that the commutator is right.

$$[\tilde{z}_x, \tilde{h}_\alpha] = [\cos \alpha \tilde{z} - \sin \alpha \tilde{h}, \sin \alpha \tilde{z} + \cos \alpha \tilde{h}] \\ = \cos^2 \alpha [\tilde{z}, \tilde{h}] - \sin^2 \alpha [\tilde{h}, \tilde{z}] = [\tilde{z}, \tilde{h}]$$