

# Fys 4110 Midterm exam 2017 Solutions

## Problem 1.

9) Hamiltonian:  $H = -\frac{\hbar \omega_0}{2} \sigma_z - \frac{\hbar \omega_1}{2} (\cos \omega t \sigma_x - \sin \omega t \sigma_y)$

We transform to a rotating frame with angular velocity  $\omega$  (same as driving field).

Time dependent unitary transform  $T(t) = e^{-\frac{i\omega}{2}t \sigma_2}$

Transformed state  $|1\rangle' = T(t)|1\rangle$

Hamiltonian  $H' = THT^\dagger + i\hbar \frac{d}{dt}T^\dagger$

Using the relations

$$e^{-\frac{i\omega}{2}t \sigma_2} \sigma_x e^{i\frac{i\omega}{2}t \sigma_2} = \cos \omega t \sigma_x + \sin \omega t \sigma_y$$

$$e^{-i\frac{i\omega}{2}t \sigma_2} \sigma_y e^{i\frac{i\omega}{2}t \sigma_2} = \cos \omega t \sigma_y - \sin \omega t \sigma_x$$

we get a time-independent Hamiltonian

$$H' = \frac{1}{2}(\omega - \omega_0) \sigma_z - \frac{\hbar \omega_1}{2} \sigma_x$$

Define:  $\Omega = \sqrt{(\omega - \omega_0)^2 + \omega_1^2}$

$$\cos \theta = \frac{\omega_0 - \omega}{\Omega} \quad \sin \theta = \frac{\omega_1}{\Omega}$$

$$H' = -\frac{\hbar}{2} \Omega \sigma_z (\cos \theta \sigma_z + \sin \theta \sigma_x)$$

This gives the time evolution

$$U'(t) = e^{-\frac{i}{\hbar} H' t} = \cos \frac{\Omega t}{2} I + i \sin \frac{\Omega t}{2} (\cos \theta \sigma_z + \sin \theta \sigma_x)$$

Transform back:  $U(t) = T(t)^\dagger U'(t) T(t)$  (2)

If  $|V(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$  with  $\underbrace{c_0(0)=1}_{\text{ground state}}$  and  $c_1(0)=0$

we get  $c_0(t) = \left(\cos \frac{\omega t}{2} + i \sin \frac{\omega t}{2} \cos \theta\right) e^{-i \frac{\omega t}{2}}$

$$c_1(t) = i \sin \frac{\omega t}{2} \sin \theta e^{i \frac{\omega t}{2}}$$

The probability to find the excited state

is  $P_1(t) = |c_1(t)|^2 = \sin^2 \frac{\omega t}{2} \sin^2 \theta$

b) Hamiltonian:  $H = \underbrace{\frac{1}{2} \hbar \omega_0 \sigma_2}_H + \underbrace{\hbar \omega_0 \sigma_3}_H + i \hbar \lambda (\sigma_- - \sigma_+)$

The eigenstates of  $H$ :  $H|I, n\rangle = \underbrace{\hbar(\omega_0 \pm \frac{1}{2}\omega_0)}_{E_{I,n}} |I, n\rangle$

The ground state is unaffected by interaction:  $H_0|-, 0\rangle = 0$

For the excited states we have:

$$H_1|+, n\rangle = i \hbar \lambda \sqrt{n+1} |-, n+1\rangle$$

$$H_1|-, n+1\rangle = -i \hbar \lambda \sqrt{n+1} |+, n\rangle$$

$\Rightarrow H_1$  mixes only pairs of states and the full  $H$  consists of  $2 \times 2$  blocks on the diagonal.

In the space  $\{|+, n\rangle, |-, n+1\rangle\}$  we have

$$H_n = \frac{1}{2} \hbar \begin{pmatrix} \Delta & -i g_n \\ i g_n & -\Delta \end{pmatrix} + E_n \mathbb{1}$$

$$\Delta = \omega_0 - \omega \quad g_n = 2 \lambda \sqrt{n+1} \quad E_n = (n + \frac{1}{2}) \hbar \omega$$

(3)

Defining  $\Omega_n = \sqrt{\Delta^2 + g_n^2}$   $\cos\theta_n = \frac{\Delta}{\Omega_n}$   $\sin\theta_n = \frac{g_n}{\Omega_n}$

$$H_n = \frac{1}{2} \hbar \Omega_n (\cos\theta_n \sigma_z + \sin\theta_n \sigma_y) + \epsilon_n \mathbf{1}$$

The eigenstates are  $|+\psi_n\rangle = \cos \frac{\theta_n}{2} |+,n\rangle + i \sin \frac{\theta_n}{2} |-,n+1\rangle$   
 $|-\psi_n\rangle = i \sin \frac{\theta_n}{2} |+,n\rangle + \cos \frac{\theta_n}{2} |-,n+1\rangle$

with eigenvalues  $E_n^\pm = \epsilon_n \pm \frac{1}{2} \hbar \Omega_n$

Using this we can now find the time evolution of a general state in the  $\{|+,n\rangle, |-,n+1\rangle\}$  space:

$$\begin{aligned} |\Psi(0)\rangle &= c_n^+(0) |+,n\rangle + c_n^-(0) |-,n+1\rangle \\ &= d_n^+ |\psi_n^+\rangle + d_n^- |\psi_n^-\rangle \\ &\xrightarrow{\text{time}} d_n^+ e^{-i \frac{\hbar}{\hbar} E_n^+ t} |\psi_n^+\rangle + d_n^- e^{-i \frac{\hbar}{\hbar} E_n^- t} |\psi_n^-\rangle \\ &= c_n^+(t) |+,n\rangle + c_n^-(t) |-,n+1\rangle \end{aligned}$$

With the initial state  $|-,n+1\rangle$  we have  $c_n^+(0)=0, c_n^-(0)=1$  and get  $c_n^+(t) = -e^{-i \frac{\hbar}{\hbar} \Omega_n t} \sin\theta_n \sin \frac{\theta_n t}{2}$   
 $c_n^-(t) = -e^{-i \frac{\hbar}{\hbar} \Omega_n t} (\cos \frac{\theta_n t}{2} + i \cos\theta_n \sin \frac{\theta_n t}{2})$

Probability for the excited state is

$$P_2(t) = |c_n^-(t)|^2 = \sin^2\theta_n \sin^2 \frac{\theta_n t}{2}$$

Comparing to the Rabi problem, this is the same provided we identify  $\omega_r \leftrightarrow \Omega_n$

9) We have  $|A(t)\rangle = C_n^+(t)|+,n\rangle + C_n^-(t)|-,n+1\rangle$   
with  $C_n^\pm(t)$  given in b).

Density matrix:  $\rho = |A(t)\rangle\langle A(t)|$

$$= |C_n^+(t)|^2|+,n\rangle\langle +,n| + C_n^+(t)C_n^-(t)^*|+,n\rangle\langle -,n+1|$$

$$+ C_n^-(t)^*C_n^-(t)|-,n+1\rangle\langle +,n| + |C_n^-(t)|^2|-,n+1\rangle\langle -,n+1|$$

Tracing over the photon mode:

$$\rho_{LS} = \text{Tr}_{\text{photon}} \rho = \sum_m \langle m | \rho | m \rangle = |C_n^+(t)|^2|+,n\rangle\langle +,n| + |C_n^-(t)|^2|-,n+1\rangle\langle -,n+1|$$

$$\text{We have } |C_n^+(t)|^2 = \sin^2 \theta_n \sin^2 \frac{\Delta n t}{2} = p^+$$

$$|C_n^-(t)|^2 = 1 - \sin^2 \theta_n \sin^2 \frac{\Delta n t}{2} = p^-$$

Entanglement entropy:

$$S = -\text{Tr } \rho_{LS} \ln \rho_{LS} = -p^+ \ln p^+ - p^- \ln p^-$$

$$= -\sin^2 \theta_n \sin^2 \frac{\Delta n t}{2} \ln \left( \sin^2 \theta_n \sin^2 \frac{\Delta n t}{2} \right)$$

$$- \left( 1 - \sin^2 \theta_n \sin^2 \frac{\Delta n t}{2} \right) \ln \left( 1 - \sin^2 \theta_n \sin^2 \frac{\Delta n t}{2} \right)$$

Maximal entropy when  $p^+$  and  $p^-$  are as equal as possible.

If  $\sin^2 \theta_n > \frac{1}{2}$ ,  $\theta_n > \pi/4$  we can get  $p^+ = p^- = \frac{1}{2}$

$$\text{with } S_{\max} = -\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \ln \frac{1}{2} = \ln 2$$

(5)

This happens when  $\sin^2 \theta_n \sin^2 \frac{\Omega n t}{2} = \frac{1}{2}$

$$\Rightarrow t = \frac{2}{\Omega n} \arcsin \left[ \frac{1}{\sqrt{2} g_n \sin \theta_n} \right] = \frac{2}{\Omega n} \operatorname{arcsinh} \left[ \frac{\sin \theta_n}{\sqrt{2} g_n} \right]$$

If  $\sin^2 \theta_n < \frac{1}{2}$  we have  $p^+ < \frac{1}{2}$  and maximal when  $\frac{\Omega n t}{2} = \frac{\pi}{2} + m\pi \quad (m \in \mathbb{Z})$

$$p_{\max}^+ = \sin^2 \theta_n \quad , \quad S_{\max} = -\sin^2 \theta_n \ln \sin^2 \theta_n - \cos^2 \theta_n \ln \cos^2 \theta_n$$

d) For the Rabi model (in rotating frame):

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

$$c_0(t) = \cos \frac{\Omega t}{2} + i \sin \frac{\Omega t}{2} \cos \theta \quad c_1(t) = i \sin \frac{\Omega t}{2} \sin \theta$$

This is a pure state and the Bloch vector has components

$$m_x^R = 2 \operatorname{Re}(c_0^* c_1) = \sin \theta \sin^2 \frac{\Omega t}{2}$$

$$m_y^R = 2 \operatorname{Im}(c_0^* c_1) = \sin \theta \sin \Omega t$$

$$\begin{aligned} m_z^R &= |c_0|^2 - |c_1|^2 = \cos^2 \frac{\Omega t}{2} + \sin^2 \frac{\Omega t}{2} \cos^2 \theta - \sin^2 \frac{\Omega t}{2} \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \sin^2 \frac{\Omega t}{2} \end{aligned}$$

For the JC model we use  $S_{\text{TLS}} = \frac{1}{2} (1 + \vec{m}^{\text{JC}} \cdot \vec{\sigma})$

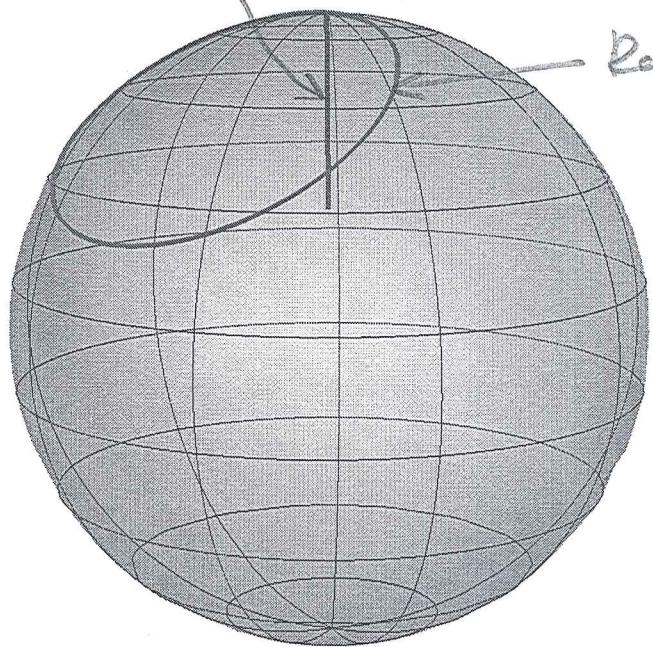
$$S_{\text{TLS}} = p^- |-\rangle \langle -| + p^+ |+\rangle \langle +| = \frac{1}{2} (1 + (p^- - p^+) \sigma_2)$$

$$\Rightarrow m_x^{\text{JC}} = m_y^{\text{JC}} = 0$$

$$m_z^{\text{JC}} = p^- - p^+ = 1 - 2 \sin^2 \theta_n \sin^2 \frac{\Omega t}{2}$$

Keynes-Cummings

Rabi



(6)

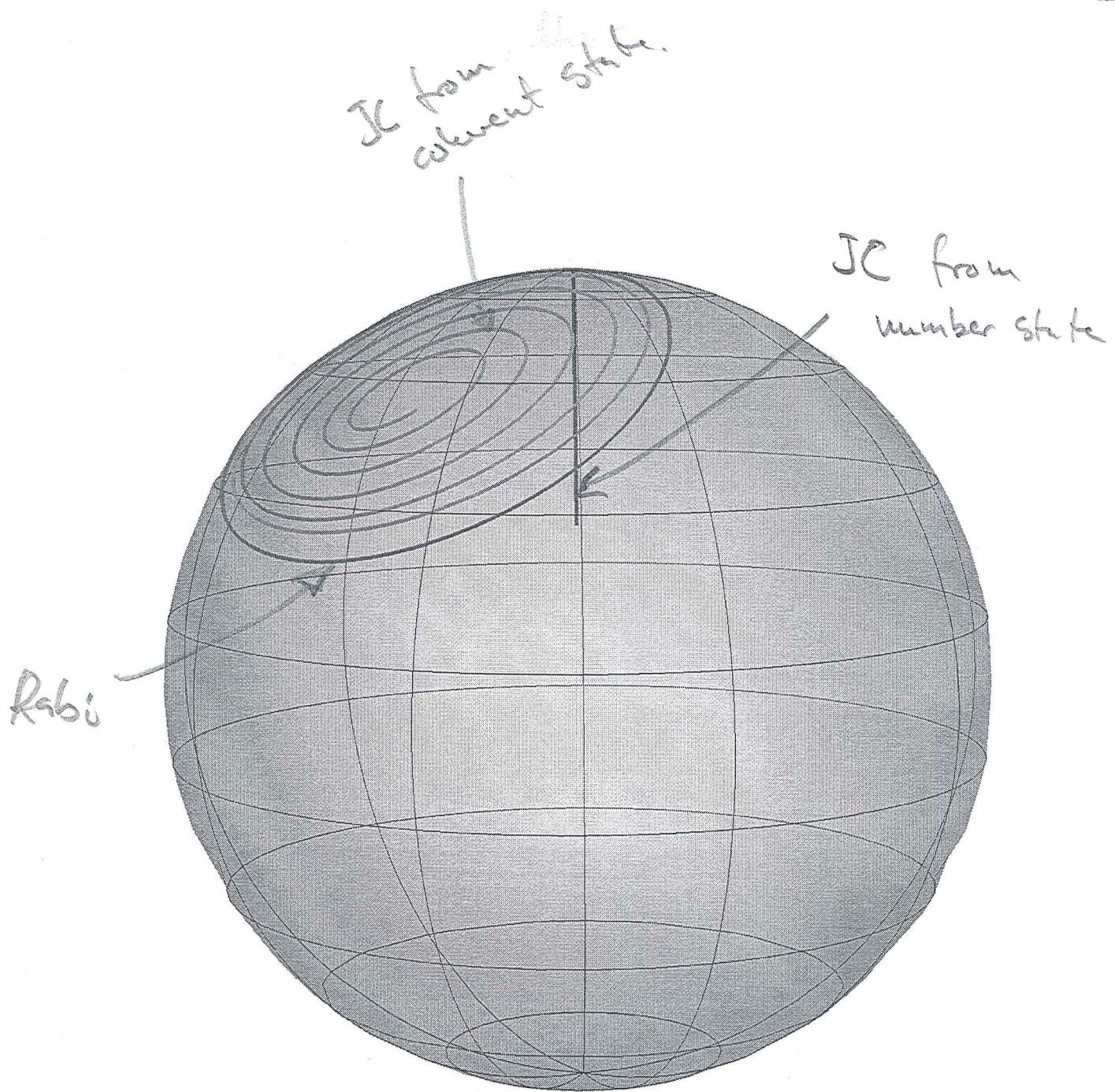
In the Rabi model, the state is always pure, and the Bloch vector precesses in a circle on the surface of the Bloch sphere.

In the JC model, the qubit is entangled with the photon mode and the reduced density matrix describes a mixed state.

The Bloch vector oscillates along the axis of the Bloch sphere with  $m_z^{JC} = m_z^R$ .

$$e) n \rightarrow \infty : \quad \Omega_n = \sqrt{\Delta^2 + g_n^2} = \sqrt{\Delta^2 + 4\lambda^2(n)} \rightarrow g_n \\ \sin \theta_n = \frac{g_n}{\Omega_n} \rightarrow 1$$

The amplitude and frequency of the oscillations decrease as  $n \rightarrow \infty$ , but the Bloch vector is always on the axis of the Bloch sphere and entanglement is not reduced. An idea for a classical limit is to assume that the photon mode starts in a coherent state instead of an eigenstate. We know that coherent states are the link to classical mechanics for the harmonic oscillator, and we can hope that it will extend to the JC model as well.



It works to some extent, but it becomes a spiral instead of circle. Here I used an average photon number of 9, maybe it should be bigger for the limit, but numerics gets slower. More work is needed...

## 7

### Problem 2

$$a) H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2} \sigma^2 + \hbar g(a^\dagger \sigma^- + a \sigma^+)$$

$| \downarrow \rangle = | 0 \rangle$   
 $| \uparrow \rangle = | 1 \rangle$

Non-interacting eigenstates:  $\{| \uparrow, n \rangle, | \downarrow, n \rangle\}$

We know that the interaction only mixes the states  $| \downarrow, n \rangle$  and  $| \uparrow, n+1 \rangle$ .

$$H |\downarrow, n \rangle = \underbrace{(\hbar\omega_r(n+\frac{1}{2}) + \frac{\hbar\Omega}{2})}_{E_{\downarrow, n}} |\downarrow, n \rangle + \hbar g \sqrt{n+1} |\uparrow, n+1 \rangle$$

$$H |\uparrow, n+1 \rangle = \underbrace{(\hbar\omega_r(n+\frac{3}{2}) - \frac{\hbar\Omega}{2})}_{E_{\uparrow, n+1}} |\uparrow, n+1 \rangle + \hbar g \sqrt{n+1} |\downarrow, n \rangle$$

$$H_n = \frac{\hbar}{2} \left( \frac{\Delta - 2g\sqrt{n+1}}{2g\sqrt{n+1} - \Delta} \right) + \hbar\omega_r(n+1) \mathbb{1} \quad \Delta = \sqrt{\Omega^2 - \omega_r^2}$$

$$\vec{E}_n = \frac{\hbar\Omega_n}{2} \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ \sin\theta_n & -\cos\theta_n \end{pmatrix} + \hbar\omega_r(n+1) \mathbb{1}$$

$$= \frac{\hbar\Omega_n}{2} (\cos\theta_n \sigma_z + \sin\theta_n \sigma_x) + \hbar\omega_r(n+1) \mathbb{1}$$

$\vec{n} \cdot \vec{\sigma}, \vec{n} = (\sin\theta_n, 0, \cos\theta_n)$

$$\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)} \quad \cos\theta_n = \frac{\Delta}{\Omega_n} \quad \sin\theta_n = \frac{2g\sqrt{n+1}}{\Omega_n}$$

$\vec{n} \cdot \vec{\sigma}$  has eigenvalues  $\pm 1$  and eigenstates

$$| +, n \rangle = \cos\theta_n |\downarrow, n \rangle + \sin\theta_n |\uparrow, n+1 \rangle$$

$$| -, n \rangle = -\sin\theta_n |\downarrow, n \rangle + \cos\theta_n |\uparrow, n+1 \rangle$$

These are also eigenstates of  $H_n$  and their eigenvalues are

$$E_{\pm n} = \pm \frac{\hbar\Omega_n}{2} + \hbar\omega_r(n+1)$$

(8)

b) For  $\Delta \gg g$  the energies are

$$E_{\pm n} = \pm \frac{\hbar \Delta}{2} \sqrt{1 + \frac{4g^2(n+1)}{\Delta^2}} \mp \hbar \omega_r (n+1)$$

$$\approx \pm \frac{\hbar \Delta}{2} \left( 1 + \frac{2g^2(n+1)}{\Delta^2} \dots \right) + \hbar \omega_r (n+1)$$

$$= (n+1) \left( \hbar \omega_r \pm \frac{\hbar g^2}{\Delta} \right) \pm \frac{\hbar \Delta}{2}$$

Level spacing:  $E_{\pm, n+1} - E_{\pm n} = \hbar \omega_r \pm \frac{\hbar g^2}{\Delta}$  independent of  $n$ .

When  $\Delta \gg g$   $\cos \theta_n \approx 1$   $\sin \theta_n \approx \frac{2g}{\Delta} \ll 1$

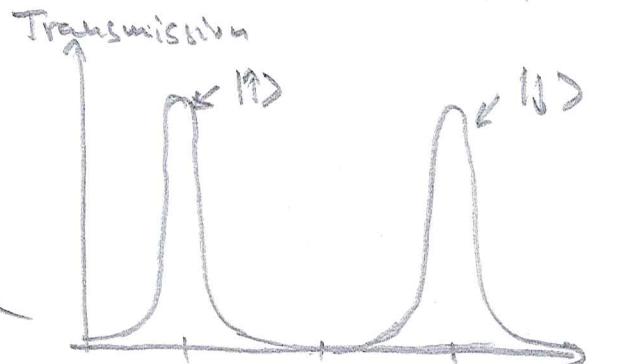
$$\Rightarrow |+\rangle \approx |n, n\rangle \quad |-\rangle \approx |1, n\rangle$$

$\Rightarrow$  Level spacing depends on qubit state.

9) The transmission is large when the microwave frequency  $\omega_{mw}$  is resonant with transitions in the system. Since

the level spacing depends on the qubit state we can determine it from the

position of the resonance



$$w_r - \frac{g^2}{\Delta}, w_r, w_r + \frac{g^2}{\Delta}, \omega_{mw}$$

line. The frequency should be chosen as one of the resonance frequencies, e.g.  $w_r - \frac{g^2}{\Delta}$

If we get large transmission amplitude at  $|1\rangle$

— o — Small —————  $\Rightarrow |0\rangle$

$$d) \text{ We use } e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] + \frac{\lambda^2}{2} [A, [A, B]] + \dots \quad (9)$$

with  $A = a\sigma^+ - a^+\sigma^-$  and  $B = H$ .

$$\text{Basic relations: } [a, a^\dagger] = 1 \quad [\sigma^\pm, \sigma^\mp] = \sigma^2$$

$$[\sigma^\pm, \sigma^2] = \mp 2\sigma^\pm$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$\sigma^+ \sigma^- = \frac{1}{2}(I + \sigma^2)$$

$$\sigma^- \sigma^+ = \frac{1}{2}(I - \sigma^2)$$

$$\sigma^+ \sigma^- + \sigma^- \sigma^+ = I$$

$$[a\sigma^+ - a^+\sigma^-, a^\dagger a] = \underbrace{[a, a^\dagger a]}_{a^\dagger \underbrace{[a, a]}_0 + \underbrace{[a, a^\dagger]}_{\frac{1}{2}} a} \sigma^+ - \underbrace{[a^\dagger, a^\dagger a]}_{a^\dagger \underbrace{[a^\dagger, a]}_{-\frac{1}{2}} + \underbrace{[a^\dagger, a^\dagger]}_0} \sigma^- = a\sigma^+ + a^+\sigma^-$$

$$[a\sigma^+ - a^+\sigma^-, \sigma^2] = a \underbrace{[\sigma^+, \sigma^2]}_{-2\sigma^+} - a^+ \underbrace{[\sigma^-, \sigma^2]}_{2\sigma^-} = -2(a\sigma^+ + a^+\sigma^-)$$

$$[a\sigma^+ - a^+\sigma^-, a^\dagger \sigma^- + a\sigma^+] = \underbrace{[a\sigma^+, a^\dagger \sigma^-]}_{a[\underbrace{\sigma^+, \sigma^-}_{\sigma^2}] + \underbrace{[a, \sigma^-]}_{a^\dagger \underbrace{[\sigma^+, \sigma^-]}_0}} - \underbrace{[a^+\sigma^-, a\sigma^+]}_{a^+ \underbrace{[\sigma^-, \sigma^+]}_{-a\sigma^2} + \underbrace{[a, \sigma^+]}_{\frac{1}{2}\sigma^2}}$$

$$= \underbrace{a a^\dagger \sigma^2}_{a a^\dagger} + \sigma^- \sigma^+ + a^\dagger a \sigma^2 + \sigma^+ \sigma^-$$

$$= (2a^\dagger a + 1) \sigma^2 + 1$$

$$[A, B] = -\hbar \Delta (a\sigma^+ + a^+\sigma^-) + \hbar g [(2a^\dagger a + 1) \sigma^2 + 1]$$

$$[A, [A, B]] = -\hbar \Delta [(2a^\dagger a + 1) \sigma^2 + 1] + \underbrace{[1]g}_{\text{Only contributes to } g^3}$$

Only contributes to  $g^3$

(10)

$$\begin{aligned}
 UHU^\dagger &\approx \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar g^2}{2} \sigma^2 + \hbar g(a^\dagger a^\dagger + a a^\dagger) \\
 &+ \frac{g}{\Delta} [-\hbar\Delta(a a^\dagger + a^\dagger a^\dagger) + \hbar g((2a^\dagger a + 1)\sigma^2 + 1)] \\
 &+ \frac{1}{2} \left(\frac{g}{\Delta}\right)^2 (-\hbar\Delta)((2a^\dagger a + 1)\sigma^2 + 1) \\
 &= \underbrace{\hbar(\omega_r + \frac{g^2}{\Delta}\sigma^2)a^\dagger a}_{\text{Level spacing.}} + \frac{\hbar}{2}(g + \frac{g^2}{\Delta})\sigma^2 + \underbrace{\frac{1}{2}\hbar\omega_r + \frac{\hbar g^2}{2\Delta}}_{\text{constant}}
 \end{aligned}$$

$$\begin{aligned}
 \sigma^2 | \downarrow \rangle &= +1 \rangle \Rightarrow \hbar\omega_r + \frac{\hbar g^2}{\Delta} \\
 \sigma^2 | \uparrow \rangle &= -1 \rangle \Rightarrow \hbar\omega_r - \frac{\hbar g^2}{\Delta}
 \end{aligned}$$

as we found in b).

e)  $H_{\mu\nu} = \hbar \epsilon (a^\dagger e^{-i\omega_{\mu\nu}t} + a e^{i\omega_{\mu\nu}t})$

$$[a^\dagger a + a^\dagger a^\dagger, a^\dagger e^{-i\omega_{\mu\nu}t} + a e^{i\omega_{\mu\nu}t}] = \sigma^+ e^{-i\omega_{\mu\nu}t} + \sigma^- e^{i\omega_{\mu\nu}t}$$

$$UH_{\mu\nu}U^\dagger \approx \hbar \epsilon (a^\dagger e^{-i\omega_{\mu\nu}t} + a e^{i\omega_{\mu\nu}t}) + \frac{\hbar g \epsilon}{\Delta} (\sigma^+ e^{-i\omega_{\mu\nu}t} + \sigma^- e^{i\omega_{\mu\nu}t})$$

f) Transformation of Hamiltonian:  $H' = THT^\dagger + i\hbar \frac{dT}{dt} T^\dagger$   
 $T = e^{i\frac{\omega_{\mu\nu}t}{2}\sigma_2 + i\omega_{\mu\nu}t a^\dagger a}$

$$\frac{dT}{dt} T^\dagger = i\omega_{\mu\nu} \left( \frac{1}{2} \sigma_2 + a^\dagger a \right)$$

T commutes with  $UHU^\dagger$

(11)

$$e^{\lambda a} a e^{-\lambda a} = a + \underbrace{\lambda [a a, a]}_{-\lambda a} + \frac{\lambda^2}{2!} \underbrace{[a a, [\lambda a, a]]}_{-\lambda^2 a} + \dots$$

$$= a - \lambda a + \frac{\lambda^2}{2!} a - \dots = e^{-\lambda} a$$

$$\Rightarrow e^{i\omega_{nw} t a^\dagger a} a e^{-i\omega_{nw} t a^\dagger a} = a e^{-i\omega_{nw} t}$$

$$e^{i\omega_{nw} t a^\dagger a} a^\dagger e^{-i\omega_{nw} t a^\dagger a} = a^\dagger e^{i\omega_{nw} t}$$

$$e^{\lambda \sigma^2} \sigma^\pm e^{-\lambda \sigma^2} = \sigma^\pm + \lambda \underbrace{[\sigma^2, \sigma^\pm]}_{\pm 2\sigma^\pm} + \frac{\lambda^2}{2!} \underbrace{[\sigma^3, [\sigma^2, \sigma^\pm]]}_{\mp \sigma^2} + \dots$$

$$= \sigma^\pm \left( 1 \pm 2\lambda + \frac{(2\lambda)^2}{2!} \pm \frac{(2\lambda)^3}{3!} \dots \right) = \sigma^\pm e^{\pm 2\lambda}$$

$$\Rightarrow e^{i\frac{\omega_{nw}}{2} \sigma^2} \sigma^\pm e^{-i\frac{\omega_{nw}}{2} \sigma^2} = \sigma^\pm e^{\pm i\omega_{nw} t}$$

$$H_{iq} = T U (H + H_{nw}) U^\dagger T^\dagger$$

$$= \hbar (\omega_r + \frac{q^2}{\Delta} \sigma^2) a^\dagger a + \frac{\hbar}{2} \left( \Omega + \frac{q^2}{\Delta} \right) \sigma^2$$

$$+ \hbar \epsilon (a^\dagger + a) + \frac{\hbar \epsilon g}{\Delta} \underbrace{(\sigma^\dagger + \sigma^-)}_{\sigma_X} - \hbar \omega_{nw} \left( \frac{1}{2} \sigma^2 + a^\dagger a \right)$$

$$= \frac{\hbar}{2} \left[ \Omega + 2 \frac{q^2}{\Delta} (a^\dagger a + \frac{1}{2}) - \omega_{nw} \right] \sigma^2 + \frac{\hbar \epsilon g}{\Delta} \sigma_X$$

$$+ \hbar (\omega_r - \omega_{nw}) a^\dagger a + \hbar \epsilon (a^\dagger + a)$$

9) With  $\omega_{\text{res}} = \omega_0 + (2n+1) \frac{g^2}{\Delta} - 2 \frac{g\varepsilon}{\Delta}$  we have

$$\omega_r - \omega_{\text{res}} = \underbrace{\omega_r - \omega_0}_{-\Delta} + (-) \frac{g^2}{\Delta} \approx -\Delta \quad \text{when } \Delta \gg g.$$

If we also assume  $\Delta \gg \varepsilon$  the term  $\hbar \varepsilon (a^\dagger a)$  will only induce small variations in the photon number  $n$ . We will ignore this term and replace  $a^\dagger a \rightarrow n$ .

$$H_{1q} = \frac{\hbar \varepsilon g}{\Delta} (\sigma_x + \sigma_z) + \text{constant.}$$

$$U(+)=e^{-\frac{i}{\hbar} H_{1q} t}$$

$$U\left(\frac{\pi \Delta}{2g\varepsilon}\right) = e^{-\frac{i\pi}{2} \frac{1}{\hbar} (\sigma_x + \sigma_z)} = \begin{pmatrix} \cos \frac{\pi}{2} & -i \sin \frac{\pi}{2} \\ i \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} = -i \cdot H$$

b) Let  $\omega_{\text{res}} = \omega_0 + \frac{g^2}{\Delta} (2n+1) \Rightarrow H_{1q} = \frac{\hbar \varepsilon g}{\Delta} \sigma_x + \text{constant}$

Rotation around  $x$ -axis with angle  $\theta$ :  $e^{-i \frac{\theta}{2} \sigma_x}$

$$\Rightarrow \frac{\varepsilon g}{\Delta} t = \frac{\theta}{2} \Rightarrow t = \frac{\Delta \theta}{2g\varepsilon}$$

$$i) H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Delta}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] \quad (13)$$

$$U = e^{\frac{g}{\Delta}[a(\sigma_1^+ + \sigma_2^+) - a^\dagger(\sigma_1^- + \sigma_2^-)]}$$

$$[A, a^\dagger a] = a(\sigma_1^+ + \sigma_2^+) + a^\dagger(\sigma_1^- + \sigma_2^-)$$

$$[A, \sigma_1^2 + \sigma_2^2] = -2[a(\sigma_1^+ + \sigma_2^+) + a^\dagger(\sigma_1^- + \sigma_2^-)]$$

$$[A, a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$$

$$= [a(\sigma_1^+ + \sigma_2^+), a^\dagger(\sigma_1^- + \sigma_2^-)] - [a^\dagger(\sigma_1^- + \sigma_2^-), a(\sigma_1^+ + \sigma_2^+)]$$

$$= 2 \left[ \underbrace{aa^\dagger}_{a^\dagger a + 1} (\sigma_1^2 + \sigma_2^2) + \underbrace{(\sigma_1^- + \sigma_2^-)(\sigma_1^+ + \sigma_2^+)}_{1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+} \right]$$

$$= 2[(a^\dagger a + \frac{1}{2})(\sigma_1^2 + \sigma_2^2) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+]$$

$$[A, H] = -\hbar\Delta[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + 2\hbar g[(a^\dagger a + \frac{1}{2})(\sigma_1^2 + \sigma_2^2) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+]$$

$$[A, [A, H]] = -\hbar\Delta 2[(a^\dagger a + \frac{1}{2})(\sigma_1^2 + \sigma_2^2) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+] + C \quad g$$

$$UHU^\dagger = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Delta}{2}(\sigma_1^2 + \sigma_2^2) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + \frac{g}{\Delta} \left\{ -\hbar\Delta[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + 2\hbar g[(a^\dagger a + \frac{1}{2})(\sigma_1^2 + \sigma_2^2) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+] \right. \\ \left. + \frac{1}{2}\left(\frac{g}{\Delta}\right)^2(-2\hbar\Delta)[(a^\dagger a + \frac{1}{2})(\sigma_1^2 + \sigma_2^2) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+] \right\} \\ = \hbar[\omega_r + \frac{g^2}{\Delta}(\sigma_1^2 + \sigma_2^2)]a^\dagger a + \frac{\hbar}{2}(2 + \frac{g^2}{\Delta})(\sigma_1^2 + \sigma_2^2) \\ + \frac{\hbar g^2}{\Delta}(\sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+) + \text{constant}$$

j) Transformation to rotating frame

$$T(t) = e^{i\frac{g^2}{2}(\sigma_1^z + \sigma_2^z) + i\omega_r t a} \quad \frac{dT}{dt} T^\dagger = i\frac{g^2}{2}(\sigma_1^z + \sigma_2^z) + i\omega_r a^\dagger$$

$T(t)$  commutes trivially with the two first terms in  $UHU^\dagger$ , and in fact also

$$\begin{aligned} [\sigma_1^z + \sigma_2^z, \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+] &= [\sigma_1^z, \sigma_1^-] \sigma_2^+ + [\sigma_1^z, \sigma_1^+] \sigma_2^- \\ &\quad + [\sigma_2^z, \sigma_2^-] \sigma_1^+ + [\sigma_2^z, \sigma_2^+] \sigma_1^- \\ &= -2\sigma_1^- \sigma_2^+ + 2\sigma_1^+ \sigma_2^- - 2\sigma_2^- \sigma_1^+ + 2\sigma_2^+ \sigma_1^- = 0 \end{aligned}$$

$$\begin{aligned} \text{So } H_{2q} &= TUHU^\dagger T^\dagger + i\hbar \frac{dT}{dt} T^\dagger \\ &= \frac{i\hbar g^2}{\Delta} (\sigma_1^z + \sigma_2^z)(a^\dagger a + \frac{1}{2}) + \frac{i\hbar g^2}{\Delta} (\sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+) \end{aligned}$$

b) We have shown that the two terms in  $H_{2q}$  commute

$$\Rightarrow U_{2q} = e^{-\frac{i}{\hbar} H_{2q} t} = e^{-\frac{i g^2}{\Delta} (\sigma_1^z + \sigma_2^z)(a^\dagger a + \frac{1}{2})} e^{-\frac{i g^2}{\Delta} t (\underbrace{\sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+}_A)}$$

$$A = \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \left( \begin{matrix} 0 \\ \sigma_x \end{matrix} \right) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e^{-\frac{i g^2}{\Delta} A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \left( \begin{matrix} 1 \\ -i \frac{g^2}{\Delta} t \sigma_x \end{matrix} \right) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{-\frac{i g^2}{\Delta} t \sigma_x} = \cos \frac{g^2 t}{\Delta} \cdot 1 - i \sin \frac{g^2 t}{\Delta} \cdot \sigma_x = \begin{pmatrix} \cos \frac{g^2 t}{\Delta} & -i \sin \frac{g^2 t}{\Delta} \\ -i \sin \frac{g^2 t}{\Delta} & \cos \frac{g^2 t}{\Delta} \end{pmatrix}$$

(15)

$$1) t = \frac{3\pi\Delta}{2g^2} \Rightarrow \frac{gt}{\Delta} = \frac{3\pi}{2} \Rightarrow M\left(\frac{3\pi\Delta}{2g^2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

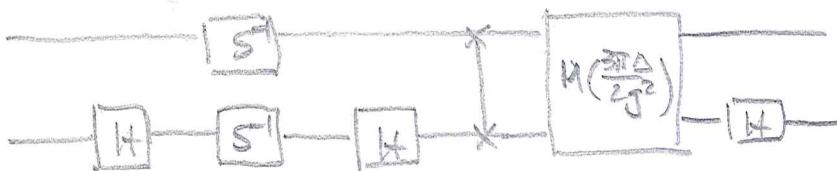
A Hadamard gate on the second qubit is

$$I \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$S^\dagger$  on both qubits is

$$S^\dagger \otimes S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The circuit



is then given by

$$\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{I \otimes H} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M\left(\frac{3\pi\Delta}{2g^2}\right)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{SWAP} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{S^\dagger \otimes S^\dagger} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{I \otimes H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = CNOT$$