FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2017

Return of solutions:
The problem set is available from Monday morning, 23 October.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday, 30 October, at 12:00.
Use candidate numbers rather than full names.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli (room V405, or on the Piazza page).

The problem set consists of 2 problems written on 5 pages.

Problem 1: Entanglement in the Jaynes Cummings model

We have in the lectures discussed Rabi oscillations of a Two Level System (TLS) driven by an external oscillating field. In this case the field is treated as a classical quantity with a given time dependence which is not affected by the dynamics of the TLS. We have also studied the Jaynes-Cummings model which is an extension of the Rabi problem to a quantized field (in a cavity, so that emitted photons are not lost, but return and can be reabsorbed). The two models gave to some extent similar results, and in this problem you are going to extend the comparison between the two models beyond what was discussed in the lecture notes or the lectures.

a) We begin by recalling the main features that we have derived. Describe the solution of Rabi problem. Sketch the derivation of these results. You do not have to repeat the full calculations, but give sufficient information so that a person familiar with the concepts will recall the arguments even if considerable time has passed since she studied it.

b) Do the same for the Jaynes-Cummings model. In particular, if we assume that the TLS is initially in the ground state and that there are $n+1$ photons in the cavity, what is the probability to find the TLS in the excited state as a function of time? Show that by a suitable mapping of the parameters, one can identify this with the solution of the Rabi problem.

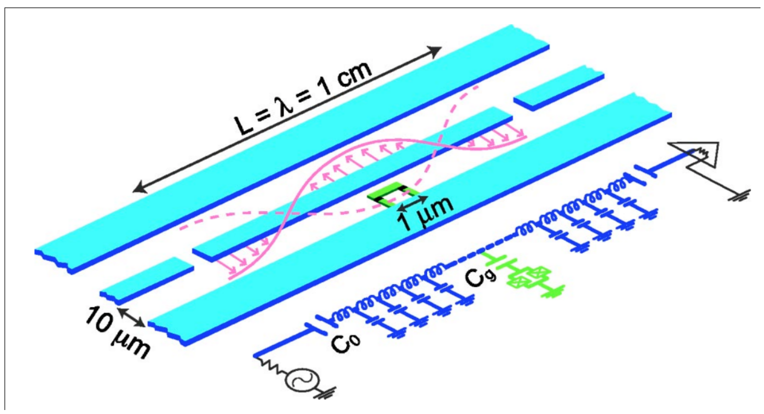
c) If we study the situation in more detail, we will see that there are differences between the two models. Assume that the initial state of the TLS is the ground state and that there are $n+1$ photons in the cavity. Find the reduced density matrix of the TLS as a function of time. Find the entanglement entropy as a function of time. What is the maximal entanglement for given parameters and when is the state maximally entangled?

d) Find the Bloch vector for the state both for the Rabi problem and the Jaynes-Cummings mode. Draw the motion of the Bloch vector in the Bloch sphere and compare the two. Describe the differences between the two models.

e) We usually think that quantum physics should approach classical in the limit where the energy of the system is much larger than the level spacing, which in this case means in the limit $n \to \infty$ where the number of photons is large. Consider you results in this limit, and discuss to what extent we have a reasonable classical limit in this case. Do you have any ideas for what could be changed to make the behaviour more classical-like in certain limit? No calculations are expected to answer this point.

Problem 2: Manipulation and readout of a superconducting qubit in a cavity

In this problem we are going to study a superconducting qubit placed inside a microwave cavity.



The cavity is a 1D transmission line resonator, which consists of a full-wave section of superconducting coplanar waveguide. A Cooper-pair box qubit is placed between the superconducting lines and is capacitively coupled to the center trace at a maximum of the voltage standing wave, yielding a strong electric dipole interaction between the qubit and a single photon in the cavity. Further details can be found in A. Blais *et al.*, Phys. Rev. A **69**, 062320 (2004).

The system is described by the usual Jaynes-Cummings model.

$$H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+)$$

where ω_r is the frequency of the cavity mode, $\hbar\Omega$ is the energy splitting of the qubit and g is the interaction strength. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the qubit ground and excited states, and $|n\rangle$ be the state of the cavity with n photons. In the noninteracting case, the eigenstates of the system are then of the form $|\uparrow, n\rangle$ and $|\downarrow, n\rangle$.

a) Find the energy eigenvales and the eigenstates of the Hamiltonian.

b) Consider in particular the case when the detuning $\Delta = \Omega - \omega_r \gg g$ and show that to second order in g, the level separation is independent of n, but depends on the state of the qubit. Find the level separation for the two qubit states $|\uparrow\rangle$ and $|\downarrow\rangle$.

c) Photons can be added to the cavity by sending external microwaves in at the end of the cavity. They will interact with the mode in the cavity by capacitive coupling through the two gaps in the central conductor in the figure. If the frequency of the external microwaves is resonant with a transition between eigenstates of the system, the coupling will be efficient, and this will result in transmission of microwaves through the system. Otherwise, most of the microwave photons will be reflected, and transmission will be small. Explain how this can be used to read out the qubit state, and specify which frequency you would use to have good discrimination between the qubit states.

d) We can obtain the same result for the state-dependent energy shift of the cavity states by a different method which will be useful in the following. Consider the unitary transform

$$U = e^{\frac{g}{\Delta}(a\sigma^+ - a^\dagger\sigma^-)} \tag{1}$$

Show that to second order in g, the transformed Hamiltonian is (here we have omitted some constant terms, which can be removed by a shift in the zero of energy).

$$UHU^\dagger \approx \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma^z\right)a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)\sigma^z \tag{2}$$

Compare with the result in b) and confirm that the resulting frequency shift is the same.

e) We can also use microwaves to manipulate the qubit. This is described by adding a term

$$H_{\mu w} = \hbar\epsilon(t)\left(a^\dagger e^{-i\omega_{\mu w}t} + ae^{i\omega_{\mu w}t}\right)$$

where $\omega_{\mu w}$ is the microwave frequency and $\epsilon(t)$ is the amplitude. It is time dependent to indicate that the microwaves will be turned on and off, and with possibly varying amplitude to achieve the desired manipulation of the qubit state. Show that to first order in g the transformed Hamiltonian is

$$UH_{\mu w}U^\dagger \approx \hbar\epsilon(t)\left(a^\dagger e^{-i\omega_{\mu w}t} + ae^{i\omega_{\mu w}t}\right) + \frac{\hbar g\epsilon(t)}{\Delta}\left(\sigma^+ e^{-i\omega_{\mu w}t} + \sigma^- e^{i\omega_{\mu w}t}\right) \tag{3}$$

We do not include second order terms, as they are of the form $\frac{g^2}{\Delta^2}\epsilon(t)$ which are small compared to the second order terms in (2) provided $\epsilon(t) \ll \Delta$.

f) To simplify the analysis, it is useful to apply a time dependent unitary transformation $T(t)$ to the system so that the Hamiltonian in the transformed representation is time-independent. This is achieved by going to a rotating reference frame both in the qubit and cavity mode. Determine the proper form of $T(t)$ and show that the resulting Hamiltonian (including both the qubit and cavity mode part (2) and the microwave driving (3)) takes the form

$$H_{1q} = \frac{\hbar}{2}\left[\Omega + 2\frac{g^2}{\Delta}\left(a^\dagger a + \frac{1}{2}\right) - \omega_{\mu w}\right]\sigma^z + \hbar\frac{g\epsilon}{\Delta}\sigma^x + \hbar(\omega_r - \omega_{\mu w})a^\dagger a + \hbar\epsilon(a^\dagger + a)$$

g) Show that if we choose the microwave frequency $\omega_{\mu w} = \Omega + (2n+1)\frac{g^2}{\Delta} - 2\frac{g\epsilon}{\Delta}$ (where n is the number of photons in the cavity) and $t = \frac{\pi\Delta}{2\sqrt{2}g\epsilon}$ the action of the microwave pulse is to perform a Hadamard gate (up to a phase) on the qubit.

h) Determine $\omega_{\mu w}$ and t so that a rotation around the x-axis with angle θ is performed.

i) We can also put two qubits inside the same cavity and the cavity can then be used to make a two-qubit gate which will entangle the two qubits. If the two qubits have the same frequency Ω, which is not resonant with the cavity frequency ω_r the Hamiltonian will be of the form

$$H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$$

Generalize the transformation (1) to the case of two qubits and show that it generates a two qubit interaction (again dropping constant terms):

$$UHU^\dagger \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}(\sigma_1^z + \sigma_2^z)\right]a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)(\sigma_1^z + \sigma_2^z) + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)$$

j) Show that in a reference frame rotating at the qubit frequency Ω in the qubit space and ω_r in the cavity mode space, the Hamiltonian takes the form

$$H_{2q} = \frac{\hbar g^2}{\Delta}(\sigma_1^z + \sigma_2^z)(a^\dagger a + \frac{1}{2}) + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)$$

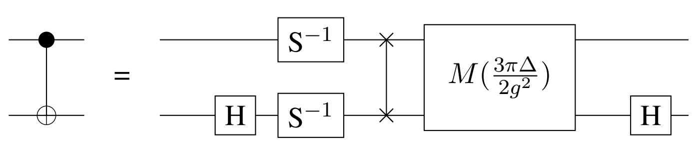
k) Show that this gives the time evolution

$$U_{2q}(t) = e^{-\frac{i}{\hbar}H_{2q}t} = e^{-i\frac{g^2}{\Delta}(\sigma_1^z+\sigma_2^z)(a^\dagger a+\frac{1}{2})}M(t)\otimes 1_r$$

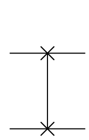
where 1_r is the unit operator in the cavity mode space and

$$M(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{g^2t}{\Delta} & -i\sin\frac{g^2t}{\Delta} & 0 \\ 0 & -i\sin\frac{g^2t}{\Delta} & \cos\frac{g^2t}{\Delta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

l) Since the matrix $M(t)$ is the only place where interactions between the two qubits enter, this is the only place where entanglement is created. To show that our system can perform universal quantum computation we need to show that it can perform the CNOT gate (which we know is universal together with one-qubit operations) using the entangling operation $M(t)$ and operations on individual qubits. Confirm that the following quantum circuit will generate the CNOT gate.



Where the SWAP gate

 $=$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

exchanges the two qubit states. It can be implemented by physically exchanging the two qubits, or by known operations with the operation $M(t)$. Here $S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ is the inverse of the phase gate and $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate.