Problem set 11

11.1 Photon emission

A particle with mass m and charge e is trapped in a one-dimensional harmonic oscillator potential, with the motion restricted to the z-axis. The frequency of the oscillator is ω. At time $t = 0$ the particle is excited to energy level n and it then performs a transition to level $n-1$ by emitting one photon of energy $\hbar\omega$. We write the energy eigenstates of the composite system of charged particle and photons as $|n, n_{\mathbf{k}a}\rangle$. With initially no photon present the state is $|i\rangle = |n, 0\rangle$, while the final state with one photon present is $|f\rangle = |n-1, 1_{\mathbf{k}a}\rangle$. To first order in perturbation theory the angular probability distribution $p(\theta, \phi)$ of the emitted photon is

$$p(\theta,\phi) = \kappa \sum_a |\langle n-1, 1_{\mathbf{k}a}|\hat{H}_{emis}|n,0\rangle|^2 \tag{1}$$

with (θ, ϕ) as the polar angle of the photon quantum number $\mathbf{k}$ and κ as a proportionality factor. $\hat{H}_{emis}$ is the emission part of the interaction Hamiltonian, which in the dipole approximation is

$$\hat{H}_{emis} = -\frac{e}{m}\sum_{\mathbf{k}a}\sqrt{\frac{\hbar}{2V\epsilon_0\,\omega}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}a}\,\hat{a}^\dagger_{\mathbf{k}a} \tag{2}$$

a) Show that for an arbitrary (real) vector $\mathbf{a}$ we have the identity

$$\sum_a (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}a})^2 = \mathbf{a}^2 - (\mathbf{a}\cdot\frac{\mathbf{k}}{k})^2 \tag{3}$$

b) Determine the particle matrix element $\langle n-1|\hat{\mathbf{p}}|n\rangle$.

c) Find the probability distribution $p(\theta, \phi)$.

The relation between the momentum operator and the ladder operators of the harmonic oscillator is found in Sect. 1.4.4 of the lecture notes.

11.2 Electric dipole transition in hydrogen (Exam 2008)

We consider the transition in hydrogen from the excited 2p level to the ground state 1s, where a single photon is emitted. The initial atomic state (A) we assume to have $m = 0$ for the z-component of the orbital angular momentum, so that the quantum numbers of this state are $(n, l, m) = (2, 1, 0)$, with n as the principle quantum number and l as the orbital angular momentum quantum number. Similarly the ground state (B) has quantum numbers $(n, l, m) = (1, 0, 0)$. When expressed in polar coordinates the wave functions of the two states (with intrinsic spin of the electron not included) are given by

$$\begin{aligned}
\psi_A(r,\phi,\theta) &= \frac{1}{\sqrt{32\pi a_0^3}}\cos\theta\,\frac{r}{a_0}\,e^{-\frac{r}{2a_0}} \\
\psi_B(r,\phi,\theta) &= \frac{1}{\sqrt{\pi a_0^3}}e^{-\frac{r}{a_0}}
\end{aligned} \tag{4}$$

where a_0 is the Bohr radius.

We remind you about the form of the interaction matrix element in the dipole approximation,

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_{emis}|A, 0\rangle = ie\sqrt{\frac{\hbar\omega}{2V\epsilon_0}}\,\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{r}_{BA} \tag{5}$$

where e is the electron charge, $\mathbf{k}$ is the wave vector of the photon, a is the polarization quantum number, ω is the photon frequency and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector. V is a normalization volume for the electromagnetic wave functions, ϵ_0 is the permittivity of vacuum and $\mathbf{r}_{BA}$ is the matrix element of the electron position operator between the initial and final atomic states.

a) Explain why the x- and y-components of $\mathbf{r}_{BA}$ vanish while the z-component has the form $z_{BA} = \nu a_0$, with ν as a numerical factor. Determine the value of ν. (A useful integration formula is $\int_0^\infty dx\, x^n\, e^{-x} = n!$.)

b) To first order in perturbation theory the interaction matrix element (5) determines the direction of the emitted photon, in the form of a probability distribution $p(\phi, \theta)$, where (ϕ, θ) are the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi, \theta)$ from the above expressions.

c) The life time of the 2p state is $\tau_{2p} = 1.6\cdot 10^{-9}s$ while the excited 2s state (with angular momentum $l = 0$) has a much longer life time, $\tau_{2s} = 0.12s$. Do you have a (qualitative) explanation for the large difference?

11.3 A radiation problem (Exam 2011)

We consider a one-dimensional problem where a two-level system (A) interacts with a scalar radiation field (B). The notation we use is similar to that in Problem 9.2. The Hamiltonian of the system we consider is

$$\hat{H} = \frac{1}{2}\hbar\omega_A\,\sigma_z + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \kappa\sum_k\sqrt{\frac{\hbar}{2L\omega_k}}(\hat{a}_k\sigma_+ + \hat{a}_k^\dagger\sigma_-) = \hat{H}_0 + \hat{H}_{int} \tag{6}$$

The first term is the two-level Hamiltonian, with energy splitting $\hbar\omega_A$, the second one is the free field contribution, with $k = 2\pi n/L$ (n - integer) as the wave number of the photon. L is a (large) normalization length. The third term is the interaction term $\hat{H}_{int}$, with κ as an interaction parameter. The frequency parameter is $\omega_k = c|k|$.

a) A general state of the two-level system is characterized by a vector $\mathbf{r}$, with $r \leq 1$, and with the corresponding density matrix as

$$\rho_A = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z\end{pmatrix} \tag{7}$$

Consider first that the interaction term $\hat{H}_{int}$ is turned off, $\kappa = 0$, so that the time evolution operator of the two-level system is $\hat{\mathcal{U}}(t) = \exp(-\frac{i}{2}\omega_A\, t\,\sigma_z)$. Use this to determine the the density matrix $\rho_A(t)$ at time t, assuming that $\rho_A(0)$ is identical to the density matrix in (7), and show that the time evolution of $\mathbf{r}$ is a precession around the z-axis with angular velocity ω_A.

b) Assume next that $\kappa \neq 0$ and that initially the two-level system is in the excited "spin up state", while the scalar field is in the vacuum state. Thus, the initial state is $|+,0\rangle = |+\rangle \otimes |0\rangle$. It decays to the "spin down state" by emission of a field quantum. The final state we then write as $|-,1_k\rangle = |-\rangle \otimes |1_k\rangle$.

The occupation probability of the excited state $|+\rangle$ decays exponentially, $P_+(t) = \exp(-\gamma t)$, with a decay rate γ that to first order in the interaction, and in the limit $L \to \infty$, is given by

$$\gamma = \frac{L}{(2\pi\hbar)^2} \int dk\, |\langle -, 1_k|\hat{H}_{int}|+,0\rangle|^2\, \delta(\omega_k - \omega_A) \tag{8}$$

Determine the decay rate γ, expressed in terms of the parameters of the problem.

As discussed in the lectures an approximate way to handle the decay is to introduce an imaginary contribution to the energy of the decaying state. Assuming a more general initial state, of the form

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle) \otimes |0\rangle = \alpha|+,0\rangle + \beta|-,0\rangle \tag{9}$$

with α and β as unspecified coefficients, with $|\alpha|^2 + |\beta|^2 = 1$, we make the corresponding *ansatz* for the time evolved state

$$|\psi(t)\rangle = (e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle) \otimes |0\rangle + \sum_k c_k(t)|-,1_k\rangle \tag{10}$$

with $c_k(t)$ as decay parameters, which satify $c_k(0) = 0$.

c) Check what normalization of the state vector (10) means for the decay parameters, and determine the reduced density matrix matrix $\rho_A(t)$ of the two-level system.

d) Assume the same initial conditions as in b), $z(0) = 1$, $x(0) = y(0) = 0$ ($\alpha = 1$, $\beta = 0$). Determine the density matrix $\rho_A(t)$ and the corresponding time dependent vector $\mathbf{r}(t)$. Is the time evolution consistent with the expected exponential decay of the excited state of the two-level system? Give a brief description of the evolution of the entanglement between the two level system and the radiation field during the decay.

e) Choose another initial condition $x(0) = 1$, $y(0) = z(0) = 0$ ($\alpha = \beta = 1/\sqrt{2}$), and find also in this case the time evolution of the reduced density matrix and the components of the vector $\mathbf{r}(t)$. Sketch the time evolution of $\mathbf{r}(t)$ and compare qualitatively the motion with that in a) and d). Find $r(t)^2$ expressed as a function of γt, and sketch also this function. What does it show about the time evolution of the entanglement between the two subsystems A and B?

Assume in this paragraph $\gamma << \omega_A$.