Problem set 2

2.1 Heisenberg's equation of motion

In the Heisenberg picture the state vectors are time independent, while the observables change with time. An observable $\hat{A}$ which has no *explicit* time dependence satisfies Heisenberg's equation of motion, in the form

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] \tag{1}$$

with $\hat{H}$ as the Hamiltonian of the system.

Assume a particle of mass m moves in a one-dimensional potential $V(x)$. In the coordinate representation the position and momentum operators are given as

$$\hat{x} = x\,, \quad \hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{2}$$

Find the expressions for Heisenberg's equation of motion for $\hat{x}$ and $\hat{p}$ and show that these give for the position operator a differential equation with the same form as the classical equation of motion of a particle with mass m in the potential $V(x)$.

2.2 Time dependent unitary transform

Two unitarily equivalent descriptions of a quantum system are related by a *time dependent* unitary transformation $\hat{U}(t)$, which acts on state vectors as

$$|\psi(t)\rangle \quad \rightarrow \quad |\psi'(t)\rangle = \hat{U}(t)|\psi(t)\rangle \tag{3}$$

and on the observables as

$$\hat{A} \quad \rightarrow \quad \hat{A}'(t) = \hat{U}(t)\,\hat{A}\,\hat{U}(t)^{-1}\,. \tag{4}$$

Show that the Hamiltonian $\hat{H}'$, which determines the Schrödinger equation of the transformed state vector $|\psi'(t)\rangle$, includes an additional term which depends on the time derivative of $\hat{U}(t)$,

$$\hat{H} \rightarrow \hat{H}'(t) = \hat{U}(t)\,\hat{H}\,\hat{U}(t)^{-1} + i\hbar\frac{d\hat{U}}{dt}\hat{U}^{-1} \tag{5}$$

Discuss the meaning of the difference between the equations (4) and (5).

2.3 Gaussian integrals

The following formula gives the integral of a gaussian function

$$I \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \tag{6}$$

This is correct for complex λ provided the real part of λ is positive. Verify this by evaluating the square I^2 as a two-dimensional integral

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,e^{-\lambda(x^2+y^2)} \tag{7}$$

and by changing to to polar coordinates in the evaluation.

Determine also the integral

$$I' \equiv \int_{-\infty}^{\infty} dx\,e^{-\lambda x^2+ax+b} \tag{8}$$

with two additional parameters, a and b.

2.4 Path integral for free particle

We will make a direct calculation of the propagator for a free particle. Start from the discretized path integral, Eq (1.101) in the lecture notes, with the potential term $V(x) = 0$. We are going to calculate each of the integrals succesively.

a) Show first that for the terms containing x_1 we have

$$I_1 = N_{\Delta t}^2 \int dx_1 e^{\frac{im}{2\hbar\Delta t}\left[(x_1-x_i)^2+(x_2-x_1)^2\right]} = \sqrt{\frac{m}{2\pi i\hbar\cdot 2\Delta t}}\,e^{\frac{im}{2\hbar\cdot 2\Delta t}(x_2-x_i)^2}$$

b) Multiply by the remaining term containing x_2 and show that

$$\begin{aligned} I_2 &= N_{\Delta t}\int dx_2 e^{\frac{im}{2\hbar\Delta t}(x_3-x_2)^2} I_1 = N_{\Delta t}\sqrt{\frac{m}{2\pi i\hbar\cdot 2\Delta t}}\int dx_2 e^{\frac{im}{2\hbar\Delta t}\left[\frac{1}{2}(x_2-x_i)^2+(x_3-x_2)^2\right]} \\ &= \sqrt{\frac{m}{2\pi i\hbar\cdot 3\Delta t}}\,e^{\frac{im}{2\hbar\cdot 3\Delta t}(x_3-x_i)^2} \end{aligned}$$

Notice how this is similar to the previous step, only with 3 replacing 2 in several places. This pattern will continue for the following steps.

c) Using this pattern prove/guess the final result after all $n-1$ integrals and compare to the result (1.109) in the lecture notes.

2.5 Path integral for harmonic oscillator

We will calculate the propagator for a harmonic oscillator by evaluating the path integral using the same method as in was done for the free particle in Eqs (1.105)-(1.108) in the lecture notes.

a) Using the Fourier expansion (1.105) in the harmonic oscillator Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$, show that the action can be written

$$S[x(t)] = S[x_{cl}(t)] + \frac{mT}{4}\sum_n \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right] c_n^2$$

b) Evaluate the path integral in the form of integrals over the Fourier coefficients c_n as in Eq (1.107) in the lecture notes and show that the propagator is

$$\mathcal{G}(x_f t_f, x_i t_i) = N e^{\frac{i}{\hbar}S[x_{cl}(t)]} \prod_n \left[1 - \left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}$$

Where N is a ω-independent normalization factor. To determine the normalization we can take the limit $\omega \to 0$ and compare to the result for a free particle that we found in Problem 2.4. You will also need the product formula

$$\prod_n \left(1 - \frac{a^2}{n^2}\right) = \frac{\sin a\pi}{a\pi}$$

In the end you should find that

$$\mathcal{G}(x_f t_f, x_i t_i) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}} e^{\frac{i}{\hbar}S[x_{cl}(t)]}$$

where $T = t_f - t_i$.

c) We still need the action along the classical path, prove that

$$S[x_{cl}(t)] = \frac{m\omega}{2\sin\omega T}\left[(x_f^2 + x_i^2)\cos\omega T - 2x_f x_i\right]$$

Warning: Even if this is a simple problem in classical mechanics, the calculations may be a bit long.

d) Use the classical action from the previous question in Eq (1.119) of the lecture notes and find the semiclasical propagator (1.116) for the harmonic oscillator. Compare it to the exact solution found above, and confirm that the semiclassical approximation is exact in this case, as expected since the Lagrangian is quadratic.