

Problem set 6

6.1 Entanglement

Two persons A and B communicate with the help of quantum entanglement. They share a set of pairs of particles with spins in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \quad (1)$$

where $|++\rangle = |+\rangle \otimes |+\rangle$ is a state where both particles of the pair have *spin up* in the z -direction, and similarly $|--\rangle = |-\rangle \otimes |-\rangle$ is the state where both particles have *spin down* in the z -direction.

- What is the quantity used as measure for the degree of entanglement in such a two-partite system, and what is the degree of entanglement in the given spin state?
- Assume A and B perform independent spin operations on their particles in a given pair, each operation described by a unitary operator, \hat{U}_A or \hat{U}_B . What happens to the entanglement of the two-particle system under such an operation.
- Assume A performs an ideal measurement of the spin component in the x - direction, which projects the spin to an eigenstate of the x -component of the spin operator. What happens to the entanglement in this case?

6.2 Schmidt decomposition 1

We have a system consisting of two spin- $\frac{1}{2}$ particles. For each of the following states, study the reduced density matrix of one of the particles and determine if the state is entangled or not. For the states which are not entangled, find a factorization of the state as a tensor product of one state for each particle. For the entangled states, find the Schmidt decomposition of the state.

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{2} (|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle) \\ |\psi_2\rangle &= \frac{1}{2} (|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle) \\ |\psi_3\rangle &= a_+ |\uparrow\uparrow\rangle + a_- |\uparrow\downarrow\rangle + a_- |\downarrow\uparrow\rangle + a_+ |\downarrow\downarrow\rangle \\ |\psi_3\rangle &= a_- |\uparrow\uparrow\rangle + a_+ |\uparrow\downarrow\rangle + a_+ |\downarrow\uparrow\rangle + a_- |\downarrow\downarrow\rangle \end{aligned}$$

where

$$a_{\pm} = \frac{\sqrt{3} \pm 1}{4}$$

6.3 Schmidt decomposition 2

Entanglement can occur not only between distinct particles, but also between different observables

for the same particle, like position and spin. Here we will find the Schmidt decomposition of one continuous and one discrete Hilbert space. A spin-half particle moving in one dimension is described by a two-component wave function

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \quad (2)$$

where the upper matrix position is assumed to correspond to "spin up" in the z -direction and the lower matrix position to "spin down" in the same direction. The scalar product of the two wave functions will generally be different from zero, and we write it as

$$\langle \psi_1 | \psi_2 \rangle = \int dx \psi_1^*(x) \psi_2(x) \equiv \Delta \quad (3)$$

a) The Schmidt decomposition of the two-component wave function has the form

$$\Psi(x) = c_1 \chi_1 \phi_1(x) + c_2 \chi_2 \phi_2(x) \quad (4)$$

where c_1 and c_2 are expansion coefficients, χ_1 and χ_2 are normalized, two-component spinors, and $\phi_1(x)$ and $\phi_2(x)$ are normalized, scalar (one-component) wave functions. What are the conditions that the spinors and wave functions should satisfy?

b) Assume the two wave functions of (2) are real Gaussian functions of the form

$$\psi_1(x) = N e^{-\lambda(x-x_0)^2}, \quad \psi_2(x) = N e^{-\lambda(x+x_0)^2} \quad (5)$$

Determine the normalization factor N and the overlap Δ , expressed in terms of λ and x_0 .

c) Determine the coefficients, spinors and wave functions in (4). (Since the wave function $\Psi(x)$ is real, you may assume the variables in Eq.(4) all to be real.)

6.4 Coupled two-level systems

Two coupled two-level systems A and B are described by the following Hamiltonian

$$\hat{H} = \frac{\epsilon}{2} (3 \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \lambda (\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \quad (6)$$

where the first factor in the tensor product refers to system A and the second factor to system B . In the equation we use the definition $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

- Write the Hamiltonian as a 4x4 matrix and show that two of the eigenvalues and eigenvectors are independent of λ . Introduce new variables, defined by $\epsilon = \mu \cos \theta$ and $\lambda = \mu \sin \theta$. Solve the eigenvalue problem for the remaining two-dimensional subspace and determine both the energies and eigenvectors as functions of μ and θ .
- Express the two eigenstates as 4x4 density matrices and determine the reduced density matrices for the two subsystems A and B .
- Determine the entropy of the reduced density matrices as functions of θ . For what parameter value is the entanglement of the two subsystems maximal?