Problem set 7

7.1 Hidden variables for a single spin-$\frac{1}{2}$

To better understand the range of theories covered by Bell's assumptions of local hidden variables, we want to show that it goes beyond what is covered in our best classical understanding of nature, and how this enables one to reproduce predictions of quantum machanics even in cases where they differ from calssical physics.

In the lecture notes, pages 14-15 we read the following:

> The experiment of Stern and Gerlach did not show such a continuous distribution. Instead the position of the atoms were rather strongly restricted to two spots, which according to the deflection formula would correspond to two possible measured values for the z-component of the magnetic moment,
>
> $$\mu_z = \pm\mu. \tag{1.38}$$
>
> This result cannot easily be explained within classical theory. To demonstrate this more directly, let us assume the y-component of the magnetic moment to be measured in a similar way by rotating the magnets. Since there is no preferred direction orthogonal to the beam, the possible results of measuring the component of the magnetic moment the y-direction should be the same as for the z-direction,
>
> $$\mu_y = \pm\mu. \tag{1.39}$$
>
> Let us further consider the component of the magnetic moment of μ in some rotated direction in the y, z-plane. For this component we have
>
> $$\mu_\phi = \cos\phi\mu_y + \sin\phi\mu_z \tag{1.40}$$
>
> with ϕ as the rotation angle relative to the y-axis. Again we may argue that due to rotational symmetry, the possible measured values of μ_ϕ should be the same as for μ_y and μ_z.
>
> $$\mu_\phi = \pm\mu. \tag{1.41}$$
>
> This clearly leads to a contradiction. The condition of discrete values for the components (1.38), (1.39) and (1.41) is not consistent with the decomposition (1.40) for a continuous set of angles . Within the framework of classical theory the observation of the discreteness of the components of the magnetic moment thus leads to a paradoxical situation.

So the predictions of quantum mechanics (and results of measurements) do not agree with the classical picture of spin as the components of a vector in space. But Bell allows us to introduce any number of additional hiden variables, which are neither controlled nor measured in the experiment. In this way he is able to reproduce the "paradoxical situation" in a setting that is consistent with our understanding of what a classical description of nature is, but not limited to present theories. In this problem you are to check that an explicit example of a hidden varible theory (invented by Bell) does give the correct results.

a) Let us start with the quantum result. Consider a spin-$\frac{1}{2}$ particle in the state $|\psi_n\rangle$ which is the eigenstate of $\sigma_{\mathbf{n}} = \mathbf{n}\cdot\boldsymbol{\sigma}$ with eigenvalue $+1$. Show that the expectation value of the spin measured along a different axis $\mathbf{a}$ is

$$\langle\sigma_{\mathbf{a}}\rangle = \langle\psi_{\mathbf{n}}|\sigma_{\mathbf{a}}|\psi_{\mathbf{n}}\rangle = \mathbf{a}\cdot\mathbf{n} = \cos\theta \tag{1}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{n}$. This is the quantum prediction that we have to reproduce using hidden variables.

b) Let the hidden variable be a unit vector $\boldsymbol{\lambda}$ distributed uniformly over the hemisphere $\boldsymbol{\lambda}\cdot\mathbf{n} > 0$. That is, the distribution function $P(\lambda)$ is constant on this hemisphere and zero on the other hemisphere. Let the outcome of the measurement of spin along the axis $\mathbf{a}$ be $A(\mathbf{a},\boldsymbol{\lambda}) = \operatorname{sgn}\mathbf{a}'\cdot\boldsymbol{\lambda}$ where $\mathbf{a}'$ is some unit vector that we will find below and $\operatorname{sgn} x$ is the sign function which is $+1$ for $x > 0$ and -1 for $x < 0$. Show that the average of the spin is

$$\langle A_{\mathbf{a}}\rangle = \int d\lambda P(\lambda)A(\mathbf{a},\boldsymbol{\lambda}) = 1 - \frac{2\theta'}{\pi}$$

where θ' is the angle between $\mathbf{n}$ and $\mathbf{a}'$. (Hint: Draw the unit sphere and identify the areas where $P(\lambda) > 0$ and where $A(\mathbf{a},\boldsymbol{\lambda})$ is positive and negative).

c) What unit vector $\mathbf{a}'$ do we have to choose so that the hidden variable theory will reproduce the quantum result (1)?

7.2 Hidden variables for anticorrelation of a pair of spin-$\frac{1}{2}$

a) We have two entangled spin-$\frac{1}{2}$ in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. Show that if we measure both spins along the same axis $\mathbf{a}$, the results will always be perfectly anticorrelated:

$$\left\langle\sigma^A_{\mathbf{a}}\sigma^B_{\mathbf{a}}\right\rangle = \langle\psi_{\mathbf{n}}|\sigma^A_{\mathbf{a}}\sigma^B_{\mathbf{a}}|\psi_{\mathbf{n}}\rangle = -1. \tag{2}$$

b) Let the hidden variable be a unit vector $\boldsymbol{\lambda}$ distributed uniformly over all directions. Let the outcomes of the measurement of spins A and B along the axes $\mathbf{a}$ and $\mathbf{b}$ be

$$A(\mathbf{a},\boldsymbol{\lambda}) = \operatorname{sgn}\mathbf{a}\cdot\boldsymbol{\lambda}$$
$$B(\mathbf{b},\boldsymbol{\lambda}) = -\operatorname{sgn}\mathbf{b}\cdot\boldsymbol{\lambda}$$

Show that the expectation value is

$$\langle A_{\mathbf{a}}B_{\mathbf{b}}\rangle = \int d\lambda P(\lambda)A(\mathbf{a},\boldsymbol{\lambda})B(\mathbf{b},\boldsymbol{\lambda}) = -1 + \frac{2\theta}{\pi}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. As long as $\mathbf{b} = \mathbf{a}$ this will reproduce the quantum result (2) of perfect anticorrelation.

7.3 Greenberger-Horne-Zeilinger (GHZ) version of Bell's theorem

We will study an example where quantum mechanics can not be reproduced by hidden variables even for results with no fluctuations (that is, without taking any averages). Consider three spin-$\frac{1}{2}$, which are sent to separate observers, A, B and C. The entangled state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$$

a) We define the operators

$$\Sigma_A = \sigma_x^A \sigma_y^B \sigma_y^C \qquad \Sigma_B = \sigma_y^A \sigma_x^B \sigma_y^C \qquad \Sigma_C = \sigma_y^A \sigma_y^B \sigma_x^C$$

Show that the three operators Σ_i commute with each other. Which means they have a common eigenstate.

b) Show that the state $|\psi\rangle$ is an eigenstate of all Σ_i with eigenvalue $+1$:

$$\Sigma_A|\psi\rangle = \Sigma_B|\psi\rangle = \Sigma_C|\psi\rangle = |\psi\rangle \tag{3}$$

Explain that this means that if all spins are measured, with two measured along the $y$direction and one along the x-direction, the product of all three results will always be $+1$. This means we will always get an even number of spins down along their chosen axis.

c) Consider now the operator $\Sigma = \sigma_x^A \sigma_x^B \sigma_x^C$ corresponding to measuring all spins along the x-direction. Show that

$$\Sigma|\psi\rangle = -|\psi\rangle \tag{4}$$

d) We now try to construct a hidden variable theory reproducing the above results. That means we have to find functions $A_X(\lambda) \equiv A_x$, A_y, B_x ...$= \pm 1$ giving the results of the measurement of each spin along each axis for a given value of the hidden variables. To agree with (3) these must satisfy the relations

$$A_x B_y C_y = 1 \qquad A_y B_x C_y = 1 \qquad A_y B_y C_x = 1 \tag{5}$$

and to agree with (4)

$$A_x B_x C_x = -1 \tag{6}$$

Show that it is impossible to satisfy all these equations at the same time.

7.4 Tsirelson's bound

For the CHSH inequality

$$S = A(B + B') + A'(B - B') \leq 2$$

we found a quantum state $|\psi\rangle$ and operators which gave $S = 2\sqrt{2}$. We are going to prove that this is the maximal value for S in quantum mechanics.

Let A and A' be operators on paricle A and B and B' be operators on paricle B. Then the A operators commute with the B operators. Also assume that the square of all the operators is 1 (like for normal spin-$\frac{1}{2}$ operators).

a) Show that

$$S^2 = 4 - [A, A'][B, B']$$

b) We define the norm of a Hilbert space vector as $\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$ and the sup norm of an Hilbert space operator M as

$$\|M\| = \sup_{|\psi\rangle} \left(\frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \right)$$

Where $\sup_{|\psi\rangle}$ means the supremum (largest possible value) for all different $|\psi\rangle$. For a normalized $|\psi\rangle$ this is exactly what we need to find the maximal value of S. Prove the following properties of the sup norm:

$$\|MN\| \leq \|M\| \, \|N\|$$
$$\|M + N\| \leq \|M\| + \|N\|$$

c) Use this to show that $\left\|S^2\right\| \leq 8$ which gives $\|S\| \leq 2\sqrt{2}$.