Problem set 9

9.1 Two-level systems (Exam 2009)

A quantum system is composed of two two-level systems, $\mathcal{A}$ and $\mathcal{B}$. The Hilbert space of the total system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is then of dimension four. The two subsystems are dynamically coupled, with the Hamiltonian of the full system having the form

$$\hat{H} = \frac{1}{2}\hbar\omega(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - i\hbar\lambda(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) \tag{1}$$

with σ_z and $\sigma_\pm$ as Pauli matrices, and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. $\hbar\omega$ is the splitting of the two energy levels of each of the two (uncoupled) systems, while λ is an interaction parameter for the composite system. In the tensor products we assume the first factor to act on subsystem $\mathcal{A}$ and the second factor to act on subsystem $\mathcal{B}$.

a) Show that the time dependent Schrödinger equation has a solution of the form

$$|\psi(t)\rangle = \cos(\lambda t)| + -\rangle + \sin(\lambda t)| - +\rangle \tag{2}$$

where $| + -\rangle = |+\rangle \otimes |-\rangle$ and $| - +\rangle = |-\rangle \otimes |+\rangle$, and where $\sigma_z|\pm\rangle = \pm|\pm\rangle$ for each of the subsystems. What is the expression for the corresponding density operator $\hat{\rho}(t)$, when this is written in bra-ket form?

b) The time dependent density operator can also be expressed in terms of Pauli matrices, in a similar way as in (1). Find this expression, and also find the reduced density operators $\hat{\rho}_A(t)$ og $\hat{\rho}_B(t)$, both expressed in terms of Pauli matrices (and the identity operator).

c) Give the general expression for the entanglement entropy of the composite system when the system is in a *pure* quantum state. For the particular, time dependent state (2), what is the expression?

9.2 Dressed photon states (Exam 2011)

A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are admixtures of the photon and atomic states. These are sometimes referred to as a ”dressed” photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{3}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{4}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the 2×2 identity matrix. Express the parameters Δ, $\cos\phi$, $\sin\phi$, and ϵ in terms of ω_0, ω and λ.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of ϕ, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.