A physical description of the universe is possible when it is assumed to be filled with matter and radiation which follows the known laws of physics. So far there is no observational evidence against this assumption. There are unknown components called dark matter and energy which are being explored these days, but they don’t matter much when the universe was very young. And that is the epoch we will consider in somewhat more detail here.

1 Astrophysical units

The mathematical formulas we use in cosmology take a simpler form when they are written in more appropriate units more suitable than the more common SI units we usually use. Time is measured in seconds and lengths is meters. Since one year contains $365.25 \times 24 \times 60 \times 60 = 3.156 \times 10^7$ seconds, we can also use

$$1 \text{ yr} = 3.156 \times 10^7 \text{ s}$$

as a time unit. Cosmological time scales are usually of the order of billion of years and thus it will later be convenient to use the time unit $1 \text{ Gyr} = 3.156 \times 10^{16} \text{ s}$. We will later discuss the age of the universe which today is known to be $13.7 \text{ Gyr}$.

Instead of using the meter, we will be a bit old-fashioned and use $1 \text{ cm} = 10^{-2} \text{ m}$ as the unit length. Since the velocity of light is defined to be $c = 299792458 \text{ m/s} = 3.00 \times 10^{10} \text{ cm/s}$, we can also measure distances in units of time. For instance, in one second a lightpulse moves $3.00 \times 10^{10} \text{ cm}$. This distance is called one lightsecond. A more common length unit is a lightyear which we will denote by the unit lyr. It has the magnitude $3.00 \times 10^{10} \text{ cm/s} \times 3.156 \times 10^7 \text{ s}$, i.e.

$$1 \text{ lyr} = 9.461 \times 10^{17} \text{ cm}$$

Astronomers are fond of using their own length unit called the parsec. For reasons we will not go into here, it is defined to be

$$1 \text{ pc} = 3.26 \text{ lyr}$$

Cosmological length scales are then typically of the order of millions of parsecs so that one then uses $1 \text{ Mpc} = 10^6 \text{ pc}$ as a suitable unit of length.

Formally, this unification of units comes about by taking the velocity of light to be $c = 1$. Times and lengths are then measured in the same units so that

$$1 \text{ s} = 3.00 \times 10^{10} \text{ cm}$$
We are then using so-called relativistic units suitable for physical problems involving
Einstein’s theory of relativity. The symbol \( c \) for the velocity of light will be absent
from all formulas. Energy and mass are then expressed in terms of the same units.
From the definition of the energy unit joule \( J \) or the equivalent electronvolt \( eV \) given
by

\[
1 \text{ J} = 6.242 \times 10^{18} \text{ eV}
\]  
(5)

it then follows that a mass of one gram corresponds to an amount of energy given
by

\[
1 \text{ g} = 5.61 \times 10^{32} \text{ eV}
\]  
(6)

For instance, elementary particle masses can be expressed in units of \( 1 \text{ MeV} = 10^6 \text{ eV} \). The proton mass is then \( m_p = 938.3 \text{ MeV} \) while the electron mass is \( m_e = 0.511 \text{ MeV} \).

In astrophysics many processes are governed by quantum mechanics. The formulas
will involve the Planck-Dirac constant \( \hbar = 1.055 \times 10^{-34} \text{ Js} = 6.582 \times 10^{-16} \text{ eVs} \). This
additional constant of nature can also be made to disappear in the mathematics by
introducing quantum mechanical units defined by setting \( \hbar = 1 \). Then we will have

\[
6.582 \times 10^{-16} \text{ eV} = 1 \text{ s}^{-1}
\]  
(7)

Needless to say, this enables us to measure other physical quantities in new units.
For example, the action of a system is usually measured in units of the Planck
constant. It will now be dimensionless. Energy can be expressed in units of inverse
seconds as follows from (7).

At very high energies the physical processes resulting from quantum mechanics will
also be relativistic. We can then combine these quantum mechanical units defined
by \( \hbar = 1 \) with the above relativistic units resulting from having \( c = 1 \) into new, so-
called HEP units. They are used in high energy particle physics and astrophysics.
In standard units we have

\[
hc = 197.328 \text{ MeV fm}
\]  
(8)

where \( 1 \text{ fm} = 10^{-15} \text{ m} \) is called one femtometer or fermi. Setting now \( hc = 1 \), we
find that \( 197.3 \text{ MeV} \) equals one inverse fermi. A very useful relation indeed for those
working in this field of physics! Obviously, it is equivalent to (7) when time and
lengths are measured in the same units. Similarly, a mass density measured in \( \text{g/cm}^3 \)
can then first be converted to \( \text{MeV/cm}^3 \) using (6) and then further on to \( \text{MeV}^4 \) with
the help of (8). In fact, we have the useful relation

\[
\text{GeV}^4 = 130.15 \text{ GeV/fm}^3
\]  
(9)
which only involves units typical of high energy nuclear physics.

Finally, when the system is at finite temperature $T$ the kinetic energies of particles will be set by the magnitude of $k_B T$. The inverse Boltzmann’s constant has the value $k_B^{-1} = 11605 \, \text{K/eV}$. Setting then $k_B = 1$, we can measure temperature in electronvolts eV instead of Kelvin degrees K, i.e.

$$1 \, \text{eV} = 11605 \, \text{K} \quad (10)$$

This thermodynamic choice of units also cleans up the formulas a lot. For instance, entropy will now be dimensionless.

\section{The Friedman equations}

The expansion of the universe follows from Einstein’s general theory of relativity. Smearing all the matter and radiation found in stars, galaxies and elsewhere out in a uniform way throughout the universe, the mathematics becomes relatively simple and can be summed in two differential equations. They are usually named after the Russian mathematician Alexander Friedman who derived them in 1924. We will need them in the following. Instead of an exact derivation, we will here give some non-relativistic arguments which reproduce the essential parts of the equations.

Consider a particle with mass $m$ a distance $a$ from some point in the universe we will call the center. This is some arbitrary point since there is no real center in the universe. In the radial direction it has the kinetic energy $m \dot{a}^2/2$ while it is pulled back by a gravitational force due to the mass inside the radius $a$. If $\rho$ is the uniform mass density of the universe, this mass is $M = (4\pi/3)a^3 \rho$. Since it corresponds to a potential energy of $-GMm/a$, the total energy of the particle is $E = m \dot{a}^2/2 - GMm/a$. We can now rewrite this as

$$\dot{a}^2 + k = \frac{8\pi}{3} G \rho a^2 \quad (11)$$

when we introduce $k = -2E/m$. And this is exactly Friedmans first equation! The only new ingredient we would have had if we had derived it from Einstein’s general theory, is the value of the constant $k$. It could then take only three values depending on the curvature of the universe. When $k = +1$ the universe has positive curvature and finite size, while for $k = 0$ it is Euclidean flat and infinite. Also in the case of negative curvature $k = -1$ it is infinite in extent but now with hyperbolic geometry.

If we think of the mass $m$ as a particle launched out into space from the mass $M$, it has a negative energy when $k = 1$ and cannot escape. It just reaches a maximal
height and falls back again. This corresponds to a finite universe. On the other hand, when \( k = -1 \), it has positive energy and can escape to infinity. Then we have an infinite universe. In the limiting case \( k = 0 \) the particle barely reaches infinity and the universe again has an infinite extent.

Instead of describing this mechanical problem using the energy principle, we could just as well do it with the help of Newtons second law. It relates the particle acceleration to the gravitational force acting on it, i.e. \( m\ddot{a} = -GMM/a^2 \). With the above expression for the mass \( M \), we then find \( \ddot{a}/a = -(4\pi G/3)\rho \). And again, this is Friedmans second equation as long as we can ignore the pressure \( p \) in the matter. If the matter moves non-relativistically, we expect this to be negligible - as we will later see. But for radiation we found the pressure \( p = \rho/3 \) and thus has a size similar to the mass density \( \rho \). A correct derivation including both of these contributions lead to the same equation as just derived but with the single replacement \( \rho \to \rho + 3p \).

The correct form of Friedmans second equation is therefore

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p)
\]  

Notice that in our units both the mass density \( \rho \) and the pressure \( p \) have the same dimensions.

Given an equation of state \( p = p(\rho) \) we now have two differential equations for the two unknowns \( a \) and \( \rho \). But they are not really completely independent of each
other. To show that we take the time derivative of the first equation (11) which gives

\[ 2\dot{a}\ddot{a} = \frac{8\pi}{3} G \frac{d}{da}(\rho a^2)\dot{a} \]

or \( \ddot{a} = \frac{(4\pi G/3)d(\rho a^2)/da} {a} \). Comparing this with the second equation (12) we thus must have

\[ \frac{d}{da}(\rho a^2) = -a(\rho + 3p) \]

for consistency. Multiplying both sides with \( a \) and combining terms, it then follows that

\[ \frac{d}{da}(\rho a^3) = -3pa^2 \quad (13) \]

But this simple result following directly from the Friedman equations is nothing else than the second law of thermodynamics \( dU + pdV = TdS \) for an adiabatic process defined by \( dS = 0 \). In fact, the energy in a sphere of volume \( V = (4\pi/3)a^3 \) is \( U = \rho V \) and \( dV = 4\pi a^2 da \) which then gives (13). Had we instead based our derivation on Einsteins relativistic theory of gravitation, we would have found the same result as a consequence of local conservation of energy and momentum during the expansion. Thus the expansion (or contraction) of the universe following from these equations will be such that the total entropy of the radiation and matter in the universe remains constant. Really an amazing and beautiful result! But perhaps not so surprising. Since there is nothing else outside the universe, it is really an isolated system for which the entropy should remain constant. But only as long as the expansion can be considered to be thermodynamically reversible. But is it? And could there not be other sources of entropy besides the one considered here? We will come back to these questions later.

In the above derivation the quantity \( a \) started out as a radius of some imaginary sphere in the universe. We must now drop this naive, mental picture. In a more realistic description it is a scale factor which determines the distance between points in the expanding universe. A simple illustration would be a two-dimensional universe corresponding to the surface of a balloon being blown up. The surface has a positive curvature and is finite in size. In this case the scale factor \( a \) is the radius of the balloon so that the distance \( D \) between two points in this universe is proportional to it. Thus \( D = \theta a \) where \( \theta \) is the angular separation between the two points seen from the center of the balloon. The points are fixed to the surface during the expansion and the angle \( \theta \) therefore stays fixed during the expansion of the balloon.
For our three-dimensional universe we have the same distance law \( D = \theta a \) between two points which follow the expansion. The points we talk about here are just mathematical idealizations, in reality they can be considered to be galaxies. They are therefore seen to move away from each other with a velocity \( V = \dot{D} = \theta \dot{a} \). This we can write as \( V = HD \) where

\[
H = \frac{\dot{a}}{a}
\]

is the Hubble parameter. We have thus arrived at one of the basic properties of the expanding universe, namely Hubble’s law saying that every galaxy moves away from us with a velocity proportional to the distance to the galaxy. The best value for the basic parameter \( H \) observed today when distances are measured in units of Mpc, is

\[
H_0 = 71 \pm 1 \text{ km s}^{-1} \text{ Mpc}^{-1}
\]

In relativistic units when distances are measured in seconds, this is equivalent to \( H_0 = 2.30 \pm 0.02 \times 10^{-18} \text{ s}^{-1} \) obtained using the conversion formulas at the beginning of this chapter.

3 Equations of state

The Friedman equations can only be solved when we know the equation of state \( p = p(\rho) \). Using it in (13) we can then solve for \( \rho = \rho(a) \) which tells us how the mass density varies with the scale factor. Combined with the first Friedman equation (11) we can then solve for \( a = a(t) \) which tells us how the universe expands with time.

Energy in the form of radiation has the equation of state \( p = \rho/3 \). We derived it first for photons, but it applies to all massless particles. It is also valid for massive particles when they are moving with momenta much larger than their masses. This is known as the extreme relativistic or ER limit opposite to the non-relativistic or NR limit where the momenta are much smaller than the mass of the particles.

Non-relativistic particles at temperature \( T \) is described by the ideal gas equation of state \( p = nk_B T \) when we can ignore the interactions between them. Here \( n = \rho/m \) is the particle number density when \( m \) is their common mass. Their average squared velocity follows from the equipartition principle as \( \langle v^2 \rangle = 3k_B T/m \) resulting in \( p = \rho \langle v^2 \rangle /3 \). But for NR particles \( \langle v^2 \rangle \ll 1 \) and we therefore can take \( p = 0 \) for such matter. This simple equation of state is also assumed to hold for the unknown,
dark matter even if we don’t know anything about the particles it consists of. But this conclusion should be ok as long as the known laws of physics applies to them.

Until ten years ago it was generally believed that the universe contained only radiation and matter. But since then it has become clear from astronomical observations that there is also a third component usually called dark energy. Even if energy and matter are equivalent in relativistic theories, dark matter and dark energy should not be confused with each other. While dark matter is assumed to have the above NR equation of state $p = 0$, the equation of state for dark energy is found to be $p = -\rho$. As we now will see, that has dramatic consequences.

All these three equations of state can be summarized in the simple equation

$$p = w \rho$$

(16)

where the parameter $w = 1/3$ for radiation, $w = 0$ for NR matter and $w = -1$ for dark energy. Inserted into the consistency equation (13) it gives

$$\frac{d}{da}(\rho a^3) + 3w\rho a^2 = 0$$

We easily rewrite it to obtain the form

$$\frac{d\rho}{\rho} = -3(1 + w)\frac{da}{a}$$

after expanding the derivative. Now it is straightforward to integrate from an epoch with scale parameter $a$ until today when it has the value $a_0$, giving

$$\rho(a) = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$$

(17)

where $\rho_0$ is the mass or energy density today.

For dark energy with $w = -1$ the dependency on the scale factor is seen to go away and the energy density $\rho_w(a) = \rho_{w0}$ is constant throughout the history of the universe. This is a special case and and is equivalent to the cosmological constant introduced by Einstein. He needed it in his very first cosmological model when he thought the universe was static, i.e. forever the same. When it was a few years later instead discovered by Hubble to expand, Einstein did not consider this constant any longer physically relevant. But today it is needed again to explain that this expansion is actually observed to be accelerating, i.e. $\ddot{a} > 0$. In fact, from (12) we see that $\ddot{a} < 0$ as long as the density $\rho$ and pressure $p$ are positive as they are for radiation and matter. But for dark energy $p = -\rho$ and the resulting acceleration is positive.
Matter has $w = 0$ and from (17) the density is seen to vary as
\[ \rho_m(a) = \rho_{m0} \left( \frac{a_0}{a} \right)^3 \]  
(18)
This can be understood from the simple fact the number of non-relativistic particles in a comoving volume is constant while the volume itself increases like $a^3$. Similarly, for radiation with $w = 1/3$ we find
\[ \rho_r(a) = \rho_{r0} \left( \frac{a_0}{a} \right)^4 \]  
(19)
If the radiation is made up of photons with a wavelength $\lambda$ this will be stretched out by the expansion and therefore increase $\propto a$. The corresponding frequency $\omega$ thus decreases like $1/a$. Since the energy of each photon is $\hbar \omega$, it will therefore also decrease like $1/a$. Combined with the volume increase $\propto a^3$, this explains the $1/a^4$ decrease of the radiative energy density (19). If it is in thermodynamical equilibrium with temperature $T$, it will be proportional to $T^4$ according to the Stefan-Boltzmann law. Comparing this with (19), one finds that it varies like
\[ T(a) = T_0 \frac{a_0}{a} \]  
(20)
during the expansion. Just after Big Bang the universe is therefore very hot. The photons from that epoch are now observed as cosmic microwave background radiation (CMBR) and have the temperature $T_{\gamma 0} = 2.725 \pm 0.001$ K. It is extremely isotropic in all directions, but there are tiny fluctuations in this temperature which are a factor ten thousands smaller. Their distribution across the sky observed by the satellites COBE and WMAP tells us very much about the properties of the early universe.

In the next section we will find that the scale factor decreases when we go back in the history of the universe. This is just another way of saying that it is expanding with increasing time. Thus, when we look back and consider the universe just after the Big Bang when $a \to 0$, the radiation density (19) will be dominant. In this early stage of the evolution we can therefore neglect the vacuum energy and even the NR matter energy. Only ER matter will contribute as part of the general radiation energy density.

The first Friedman equation (11) we can write as
\[ \frac{k}{a^2} = \frac{8\pi}{3} G \rho - H^2 \]
when we reorder it and introduce the Hubble parameter (14). It can be rewritten as $k/a^2 = H^2(\Omega - 1)$ when we define $\Omega = \rho/\rho_c$ where the so-called critical density is
\[ \rho_c = \frac{3H^2}{8\pi G} \]  
(21)
For its numerical value we need the gravitational constant

\[ G = 6.673 \times 10^{-8} \frac{\text{cm}^3}{\text{g s}^2} \]  

which in relativistic units is equivalent to \( G = 7.425 \times 10^{-29} \text{cm/g} \). With the present value (15) for the Hubble constant, the critical density is then found to be \( \rho_{c0} = 9.4 \times 10^{-30} \text{g/cm}^3 = 5200 \text{eV/cm}^3 \) today. It was larger in the past.

Measurements by the WMAP satellite combined with other astronomical observations have made it clear during the last few years that the universe is spatially flat. The curvature parameter is therefore \( k = 0 \) and the relative matter density \( \Omega = 1 \). Since we have three kinds of matter in the universe, we must therefore always have

\[ \Omega_r + \Omega_m + \Omega_v = 1 \]  

where \( \Omega_i = \rho_i/\rho_c \) for each component. Different observations give the present values \( \Omega_{\gamma0} = 0.73 \) and \( \Omega_{m0} = 0.27 \). Thus the dominant energy component of the universe today is the vacuum energy. Matter appears as dark matter and ordinary, baryonic matter with densities \( \rho_d \) and \( \rho_b \) respectively so that \( \rho_m = \rho_d + \rho_b \). The relative matter energy density \( \Omega_m = \Omega_d + \Omega_b \) is today dominated by the dark component with \( \Omega_{d0} = 0.23 \). Therefore \( \Omega_{d0} = 0.04 \) or only 4% of the energy content of the universe is due to the same kind of matter as we know here on Earth!

The contributions from radiation is today negligible. This follows from the Stefan-Boltzmann law which is

\[ \rho_\gamma = \frac{\pi^2}{15} T^4 \gamma \]  

in HEP units. With the present temperature \( T_{\gamma0} = 2.725 \pm 0.001 \text{K} \) of the CMB radiation, this gives \( \rho_{\gamma0} = 0.26 \text{eV/cm}^3 \). Dividing by the critical density (21), it follows that \( \Omega_{\gamma0} = 5 \times 10^{-5} \). One can show that the contribution from starlight in the universe is even smaller. Later we will see that massless neutrinos will give a contribution of the same, small magnitude. But today we know that the neutrinos are not exactly massless and thus can give a significantly bigger contribution. We will come back to this part later.

### 4 Expansion of the universe

Our universe is flat universe with curvature parameter \( k = 0 \) and the first Friedman equation (11) simplifies to \( \ddot{a}^2 = 8\pi G \rho a^2/3 \) or

\[ H^2 = \frac{8\pi G}{3} \rho \]  

9
since $H = \dot{a}/a$. Here $\rho$ is the total mass-energy density with the three principal parts
$\rho = \rho_r + \rho_m + \rho_v$. Each varies with the scale parameter according to (17). Dividing
by $H_0^2$ and expressing the present mass densities by their relative magnitudes, we then get

$$\frac{H^2}{H_0^2} = \Omega_{r0} \left( \frac{a_0}{a} \right)^4 + \Omega_{m0} \left( \frac{a_0}{a} \right)^3 + \Omega_{v0}$$

This is now a first-order differential equation for the scale parameter when we take
the square root on both sides. Writing it as

$$\frac{da}{a} \left[ \Omega_{r0} \left( \frac{a_0}{a} \right)^4 + \Omega_{m0} \left( \frac{a_0}{a} \right)^3 + \Omega_{v0} \right]^{-1/2} = H_0 dt \quad (26)$$

we see that it can be solved by direct integration. The solution $a = a(t)$ describes
the evolution of the universe.

Ten years ago it was generally believed that the vacuum energy could be ignored
so that the universe was mass dominated today. In such a case we have $\Omega_{v0} = 0$
and again neglecting the small contribution from radiation today, it follows that
$\Omega_{m0} = 1$ from (23) to keep the universe is flat. From the master equation (36) we
then integrate from an initial time $t = 0$ with $a = 0$ to a later time $t$ with scale
parameter $a$, i.e.

$$\int_0^a da \left( \frac{a}{a_0} \right)^{1/2} = H_0 \int_0^t dt$$

We have here also neglected the radiation contribution at very early times which is
not completely correct. For better accuracy it must be included. Ignoring this small
complication, the above integrals are now elementary and give for the scale factor

$$a(t) = a_0 \left( \frac{3H_0 t}{2} \right)^{2/3} \quad (27)$$

It increases with time as illustrated in Fig.2 and becomes equal to the present value
$a_0$ at a time when $3H_0 t_0 = 2$ or

$$t_0 = \frac{2}{3H_0} \quad (28)$$

With the value (15) for the Hubble constant this gives $t_0 = 9.4$ Gyr or somewhat
less than ten billion years. It would be the age of the universe and would not change
very much including the radiation part.

The problem with this simple cosmology was that the universe contains clusters of
stars being nearly thirteen billion years old. And they must be younger than the
universe! The simplest way to obtain a longer age was to include the vacuum energy. We must then repeat the above calculation leading to the somewhat more difficult integral

\[
\int_0^a \frac{da}{(Ω_v0a^2 + Ω_m0a^3/a)^{1/2}} = H_0 \int_0^t dt
\]

Here again \(Ω_m0 + Ω_v0 = 1\). It can be done using hyperbolic functions resulting in

\[
a(t) = a_0 \left(\frac{Ω_m0}{Ω_v0}\right)^{1/3} \sinh^{2/3} \left(\frac{3}{2} \sqrt{Ω_v0}H_0 t\right)
\]

Again the age of the universe \(t_0\) is determined by \(a(t_0) = a_0\) which gives

\[
t_0 = \frac{2}{3H_0 \sqrt{Ω_v0}} \text{arcsinh} \left(\frac{Ω_v0}{Ω_m0}\right)^{1/2}
\]

Inserting the measured values for the different parameters we find the more realistic age \(t_0 = 13.7\) Gyr. It is consistent with the age of the oldest globular star clusters and is now the generally accepted age of the universe.

It is easy to see from (29) that the expansion of the universe is today actually accelerating, i.e. \(\ddot{a} > 0\). It has actually been doing that for the last several billion years.
In the future when \( t \gg t_0 \) the hyperbolic function becomes a simple exponential function resulting in

\[
a(t) = a_0 \left( \frac{\Omega_m 0}{4 \Omega_m 0} \right)^{1/3} e^{\sqrt{\Omega_m 0} H_0 t} \tag{31}
\]

This type of cosmology was first studied by the Dutch astronomer de Sitter just after Einstein had applied his general theory of relativity to the universe. On the other hand, at earlier times the expansion was instead decelerating. This can be seen directly from the second Friedman equation (12) giving \( \ddot{a} < 0 \) since then \( p > 0 \) when only matter and radiation are present.

5 The radiation-dominated early universe

At very early times when \( a \to 0 \) the radiation term dominates in the equation of motion (36). It then simplifies to \( ada = a_0^2 \sqrt{\Omega_r 0 H_0} dt \) which gives the solution

\[
a(t) = a_0 \left( 2 \sqrt{\Omega_r 0 H_0} t \right)^{1/2} \tag{32}
\]
after integration. It is only valid as long as \( a \ll a_0 \). With increasing time the matter component of the universe will eventually start to matter resulting in a somewhat faster expansion of the type (27). The radiation energy density \( \rho \) will decrease as (19) as a function of the scale factor and therefore as \( \rho = \rho_{r0} / 4 \Omega_r 0 H_0^2 t^2 \) for increasing times when we use the above result for \( a(t) \). Now \( \Omega_{r0} = \rho_{r0} / \rho_{c0} \) where the critical density today is \( \rho_{c0} = 3 H_0^2 / 8 \pi G \). Thus we find the important formula

\[
\rho = \frac{3}{32 \pi G t^2} \tag{33}
\]
relating directly the density in the early universe to its age.

What we call radiation here, is not only made up of photons, but all particles with masses much smaller than the temperature \( T \) in the universe. Furthermore, we will assume that all chemical potentials are zero so that the number of particles and antiparticles are the same. We will also assume that interactions between the particles making up the radiation can be neglected. This is a highly non-trivial assumption and is related to the property that all particles are described by so-called non-Abelian gauge theories having this crucial property. The radiation energy density \( \rho \) is then simply given by the Stefan-Boltzmann formula, adjusted by the spin degrees of freedom for each particle type. Using HEP units with \( c = \hbar = 1 \) we can thus write

\[
\rho = g \frac{\pi^2}{30} T^4 \tag{34}
\]
when we also set \( k_B = 1 \), measuring temperature in energy units according to the conversion formula (10). Here we have also introduced the spin degeneracy factor which is \( g = 1 \) for each bosonic spin degree of freedom and \( g = 7/8 \) for the corresponding fermion spin degrees of freedom. For example, a photon has two possible spin directions and therefore \( g_{\gamma} = 2 \). A massless neutrino has always the spin in the opposite direction of its momentum and therefore \( g_{\nu} = 7/8 \) while an electron can have its spin both along and opposite to the momentum giving \( g_e = 7/4 \).

Combining (33) and (34) we can now express the squared age of the universe directly by its temperature,

\[
t^2 = \frac{3 \times 30}{32\pi^3 g} \frac{1}{GT^4}
\]

In the same units the gravitational constant (22) is found to have the value

\[
\sqrt{\frac{1}{G}} = 1.22 \times 10^{19} \text{GeV}
\]

so that

\[
\frac{1}{GT^4} = 1.49 \times 10^{44} \frac{\text{MeV}^2}{(T/\text{MeV})^4}\text{MeV}^4
\]

Here \( T/\text{MeV} \) is the temperature measured in units of MeV which we henceforth will denote by \( T_{\text{MeV}} \). Now from (8) we have \( 197.3 \text{ MeV} = 1.0 \text{ fm}^{-1} = 1.52 \times 10^{21} \text{ s}^{-1} \) and therefore \( 1/\text{GT}^4 = 64.5 \text{ s}^2/T_{\text{MeV}}^4 \). Using this in the above formula for the squared age of the universe, it results in \( t^2 = 5.85 \text{ s}^2/g T_{\text{MeV}}^4 \). Taking the square root and measuring the time in units of seconds, i.e. defining the dimensionless time \( t_s = t/s \), we finally have the very important formula

\[
t_s = \frac{2.42}{\sqrt{g} T_{\text{MeV}}^2}
\]

It gives directly the age of the universe measured in units of seconds, in terms of its temperature, measured in MeV. We can therefore already say that the universe must have a temperature around 1 MeV when it is approximately one second old. Notice that there is no nuclear or elementary particle physics information used in getting this result!

### 6 Formation of nucleons

All the known elementary particles are described within a theoretical framework called the standard model. It predicts that when they are in thermal equilibrium at a
temperature \( T \gg 100 \text{ GeV} \) they are all massless and will thus make up the radiation content of the early universe. Thermal equilibrium will exist when the expansion rate of the universe is slower than the reaction rate between the different particles. These rates can be calculated and we know when equilibrium thermodynamics can be used to describe their properties.

The bosonic particles are 8 colored guons, 4 electroweak vector bosons and 4 spinless Higgs particles. Since the first two groups have spin \( S = 1 \) like the photon, their contribution to the \( g \)-factor becomes

\[
g_B = (8 + 4) \times 2 + 4 = 28 \quad (37)
\]

In the fermion sector there are 6 quarks, each in three different colors, 3 leptons like the electron and 3 neutrinos. With the same contribution from their antiparticles, the fermionic \( g \)-factor becomes

\[
g_F = \frac{7}{8} \left( (6 \times 3 + 3) \cdot 2 + 3 \right) \times 2 = \frac{7}{8} \cdot 90 \quad (38)
\]

For the total \( g \)-factor \( g = g_B + g_F \) needed in the formula (36) we thus obtain \( g = 427/4 = 106.25 \). A temperature of \( T = 500 \text{ GeV} \) is then seen to correspond to an age of \( t = 10^{-12} \text{ s} \) of the universe! For even earlier times and therefore also higher temperatures, we will need improvements of the standard model. So far that is unknown territory which will hopefully be explored in the not too distant future.

Let us now sketch the thermal history as the universe evolves from these early times and cools down. When the temperature has become \( T < 100 \text{ GeV} \) most of the particles have become massive except for the photon and the gluons. But the temperature is still so high that most of them can still be considered to be massless and therefore described as radiation. Exceptions are the top quark with a mass of 174 GeV and the weak bosons with masses around 80 – 90 GeV. But these are all unstable and will decay into lighter particles. When the temperature has cooled below \( T = 1 \text{ GeV} \) the universe consists of \( u, d, s \) colored quarks and their antiparticles, gluons, photons, neutrinos and antineutrinos plus the leptons \( e^{\pm} \) and \( \mu^{\pm} \). They can all be considered to be massless as long as \( T > 200 \text{ MeV} \). The total \( g \)-factor is therefore

\[
g = (8 + 1) \times 2 + \frac{7}{8} \left( (3 \times 3 + 2) \cdot 2 + 3 \right) \times 2 = 247/4 \quad (39)
\]

or \( g = 61.75 \). In this temperature interval the universe is therefore around \( t = 10^{-6} \text{ s} \) old.

After the temperature has fallen below \( T = 100 \text{ MeV} \) the quarks, antiquarks and gluons have combined into baryons with strong interactions. Earlier there must
have been slightly more quarks than antiquarks because we know that the ordinary matter in the universe is made up of nucleons, i.e. protons \( p \) and neutrons \( n \), and not antinucleons. The nucleons are now stable particles with masses \( m_p = 938.3 \text{ MeV} \) and \( m_n = 939.6 \text{ MeV} \). They will therefore not contribute to radiation energy, but instead make up the non-relativistic baryonic matter density. The muon leptons \( \mu^\pm \) have a mass \( m_\mu = 105.7 \text{ MeV} \), but are unstable and will be replaced by the decay products \( e^\pm \), photons and the neutrinos with their antiparticles. As long as \( T > 1 \text{ MeV} \) these particles can be considered as radiation with a

\[
g = 2 + \frac{7}{8} \left( 1 \cdot 2 + 3 \right) \times 2 = 43/4 \tag{40}
\]

When the temperature has dropped down to \( T = 1 \text{ MeV} \), it follows then from (36) that the expansion of the universe has lasted very close to \( t = 1 \text{ s} \).

At these temperatures protons and neutrons are still in thermal equilibrium with the other particles through reactions like

\[
\begin{align*}
p + e^- & \leftrightarrow n + \nu_e \\
n + e^+ & \leftrightarrow p + \bar{\nu}_e
\end{align*}
\]

mediated by the weak force. Since these nucleons are non-relativistic, their number densities are given by the Maxwell distribution

\[
n_i = g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T} \tag{41}
\]

where the spin degeneracy factor \( g_i = 2 \) for these spin-1/2 fermions. Since the proton has a slightly smaller mass than the neutron, there will be somewhat more protons than neutrons in the universe at these temperatures according to the formula

\[
\frac{n_n}{n_p} = \left( \frac{m_n}{m_p} \right)^{3/2} e^{-\Delta m/T} \tag{42}
\]

where \( \Delta m = m_n - m_p = 1.29 \text{ MeV} \). Here as before we ignore any chemical potentials. This ratio will therefore decrease when the temperature sinks below \( T = 1 \text{ MeV} \) and there will thus be fewer neutrons than protons in the universe.

More detailed calculations show that when the temperature has fallen to \( T = 0.8 \text{ MeV} \), the above reactions goes out of equilibrium. The corresponding ratio (42) then has the value \( n_n/n_p = \exp(-1.29/0.8) = 0.2 = 1/5 \). One says that there is a nucluon 'frezze-out'. At this stage of the evolution there are therefore five protons for each neutron, i.e. the nucleon content of the universe is \( pppppn \). The number fraction of neutrons is therefore

\[
X_n = \frac{n_n}{n_n + n_p} = \frac{1}{1 + n_p/n_n} = \frac{1}{1 + 5} = \frac{1}{6} \tag{43}
\]
Because the universe is assumed to be electrically neutral, there will be the same number of electrons as protons.

7 Neutrino freeze-out

During this epoch the three neutrinos types $\nu_e, \nu_\mu$ and $\nu_\tau$ and their antiparticles remain in thermal equilibrium with each others through weak interactions like

$$e^+ + e^- \leftrightarrow \nu + \bar{\nu}$$
$$e^+ + \nu \leftrightarrow e^+ + \nu, \quad e^+ + \bar{\nu} \leftrightarrow e^+ + \bar{\nu}$$

The corresponding reaction rates decrease with decreasing temperatures and when $T$ falls below 1 MeV they become slower than the expansion rate of the universe. Thus the neutrinos will also go out of thermal equilibrium around this temperature. Their number densities will still be described by the Fermi-Dirac distribution, but will no longer be affected by the presence of other particles. Throughout this epoch there is also a small contribution to the energy density from the baryons and the cold dark matter, but since these particles are non-relativistic and few in number compared with the others, these contributions can be ignored.

When decreasing temperature the positrons disappear around $T = 0.5 \text{ MeV}$. Until then they are in equilibrium with the electrons and photons through reactions like $e^+ + e^- \leftrightarrow \gamma + \gamma$. For this reaction to proceed, the photons need to have an energy greater than 0.5 MeV which is the mass of the electron/positron. For lower temperatures essentially all the positrons annihilate with electrons and are converted to photons according to $e^+ + e^- \rightarrow \gamma + \gamma$. Thus the density of photons increase, corresponding to an increase in the photon temperature. This we can now calculate since we know that the total entropy $S = sV$ has to be conserved. For the volume we can take $V = (4\pi/3)a^3$ while the entropy density is $s = 4\rho/3T$ when there are no chemical potentials. From the energy density (34) we then get from the equality of entropy before and after annihilation

$$\left(2 + \frac{7}{8} \cdot 2 \cdot 2\right) \cdot (aT_\gamma)_\text{before} + \frac{7}{8} \cdot 2 \cdot 3 \cdot (aT_\nu)_\text{before}$$
$$= 2 \cdot (aT_\gamma)_\text{after} + \frac{7}{8} \cdot 2 \cdot 3 \cdot (aT_\nu)_\text{after}$$

Since the neutrinos are decoupled, we will have $(aT_\nu)_\text{before} = (aT_\nu)_\text{after}$. Thus it follows that $(aT_\gamma)_\text{after}/(aT_\gamma)_\text{before} = (11/4)^{1/3}$. But before the annihilation the neutrino and photon temperatures were the same and therefore

$$\frac{T_\gamma}{T_\nu}_\text{after} = \left(\frac{11}{4}\right)^{1/3} = 1.401$$

(44)
This ratio will now remain the same until the present epoch when $T_\gamma = 2.73$ K. Thus the temperature of the cosmic neutrino background radiation today is $T_{\nu 0} = 1.95$ K.

So far we have assumed the neutrinos to be massless and their properties given by standard Fermi-Dirac statistical mechanics. But we know today that they actually have non-zero masses of magnitudes somewhat less than 1 eV. So the above description is only valid when $T_{\nu} \gg m_{\nu}$. But for these temperatures their number density is given by $n_{\nu+\bar{\nu}} = (3/4)n_{\gamma}$ which will be valid down to temperatures $T_{\nu} < m_{\nu}$ since the neutrinos are stable particles. The number density of one neutrino flavor today is therefore

$$n_{\nu+\bar{\nu}}(t_0) = \frac{3}{4} n_{\gamma}(t_0) \cdot \frac{4}{11} = \frac{3}{11} \cdot 413 \text{ cm}^{-3} = 112 \text{ cm}^{-3}$$

(45)

and the corresponding non-relativistic mass density

$$\rho_{\nu+\bar{\nu}}(t_0) = m_{\nu} n_{\nu+\bar{\nu}}(t_0)$$

For the energy density ratio resulting from the three neutrino flavors known today we therefore have

$$\Omega_{\nu 0} = \sum_{\nu=1}^{3} \rho_{\nu+\bar{\nu}}(t_0)/\rho_{c0} = \frac{M_{\nu}}{47 \text{ eV}}$$

(46)

where $M_{\nu} = \sum_{\nu=1}^{3} m_{\nu}$. We have here used the previous value $\rho_{c0} = 5200 \text{ eV/cm}^3$ for the critical mass density today. We thus see that if the neutrinos had had somewhat larger masses, around 5-10 eV, they could have been very good candidates for dark matter!

## 8 Helium production

The lightest and most stable nucleus is found in the helium $He$ atom and is usually called an $\alpha$ particle made up of two protons and two neutrons. Above we found that the nucleon content of the universe was $ppppnpnppppp$ when the temperature had fallen below 0.8 MeV. If the $He$ nucleus could form directly from this mixture, we see from (43) that there would be one $\alpha$ particle among eight protons. The weight fraction of helium nuclei would therefore be $Y_\alpha = 4/(4 + 8) = 1/3 = 0.33$ since each weighs four times as much as a proton.

This is somewhat higher than the observed ratio 0.25. The reason is that an $\alpha$-nucleus cannot form directly from two protons and two neutrons. Instead one must first form deuterium $d$ from the reaction

$$p + n \rightarrow d + \gamma$$

(47)
where the photon has the energy \( E_{\gamma} = m_n + m_p - m_d = 2.22 \text{ MeV} \). But this intermediate production will not take place right-away for temperatures \( T < 0.8 \text{ MeV} \). The reason is that this value represents the average temperature of the blackbody photon radition. There will remain a small tail in the Planck distribution of photons with energies \( E > 2.22 \text{ MeV} \) even at these lower temperatures. And as long as that is the case, the reaction (47) will also go in the reverse direction, breaking up the deuterium nuclei.

More detailed calculations show that first when the temperature has sunk to \( T = 0.08 \text{ MeV} \) will the deuterium nuclei start to survive. And then follows very fast reactions like \( d + d \rightarrow \alpha + \gamma \) and others producing stable helium nuclei. A very small amount of deuterium will avoid this formation and survive to be observed in the universe. The radiation will now consist only of photons and the decoupled neutrinos with the temperature given by (44). For the effective g-factor we then have

\[
g = 2 + \frac{7}{8} \cdot 3 \cdot 2 \cdot \left( \frac{4}{\Pi} \right)^{4/3} = 3.36
\]

(48)

From the master formula (36) follows then that the corresponding age of the universe is

\[
t = \frac{2.42}{1.83 \cdot 0.08^2} \frac{s}{s} = 210 \text{ s}
\]

or three and one half minutes.

We can now understand why we above got a result for the fraction of helium which was a bit too large. Since free neutrons are radioactively unstable and decay like \( n \rightarrow p + e^- + \bar{\nu}_e \) with a lifetime of \( \tau = 890 \text{ s} \), the number of neutrons available for deuterium production will be reduced by the factor \( \exp(-210/890) = 0.79 \). Thus the final weight fraction of helium becomes \( Y_\alpha = 0.33 \cdot 0.79 = 0.26 \) which is very close to the observed value.

After 210 seconds evolution the main ingredients of the universe is then photons, neutrinos, electrons, protons and \( \alpha \)-particles forming a plasma with complicated, electromagnetic interactions. It will continue to be radiation dominated for a very long time and cooling down according to the formula (36). Nothing much happens and it is very dull period. Eventually the temperature will be become a few \( eV \) and neutral hydrogen atoms \( H \) will form according to

\[
p + e^- \rightarrow H + \gamma
\]

(49)

and similarly for the formation of neutral helium atoms \( He \). Earlier the photon energies very so high that these reactions also went just as likely in the opposite
direction. But from this time on there are no charged particles in the universe and the photons can move freely. This transition at time $t_{\text{dec}}$ to neutral matter is usually called decoupling or time for last scattering. The photons we see today in the CMB radiation comes from this epoch.

Assuming radiation dominance all the way to this decoupling time, we can then calculate it from the master formula (36). We know that when the temperature was $T_* \simeq 0.1 \text{ MeV}$ the age was $t_* \simeq 100 \text{ s}$. Thus it follows that

$$t_{\text{dec}} = \left( \frac{T_*}{T_{\text{dec}}} \right)^2 t_* \simeq \left( 0.1 \cdot 10^6 \right)^2 \cdot 100 \text{ s} = 10^{12} \text{ s}$$

which roughly corresponds to $t_{\text{dec}} \simeq 100,000 \text{ years}$. From then on the almost uniform matter in the universe will slowly start to clump together by the gravitational interaction, stars will form which later cluster into galaxies and so on until we have the universe we see today.

April 2010, Finn Ravndal