Exercise 1

In this exercise we will use the axiom of set existence (axiom 0)
\[ \exists x (x = x), \]
the axiom of extentionality (axiom 1)
\[ \forall x \forall y [ \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y], \]
and the comprehension scheme (axiom 3)
\[ \forall y \exists z (x \in z \leftrightarrow x \in y \land \phi(x)). \]

Part 1a

Explain why any model satisfying the above axioms must also satisfy
\[ \exists y \forall x (x \not\in y). \]

Hint: \( \phi \) can be a contradiction.

Part 1b

Explain how we can use comprehension to show that
\[ \exists ! y \forall x (x \not\in y), \]
where \( \exists ! y \psi(y) \) is equivalent to \( \exists y (\psi(y) \land \forall x (\psi(x) \rightarrow x = y)) \).

Hint: If \( y \) is empty then \( x \not\in y \) for any \( x \).
Part 1c
Since there is only one empty set, we can introduce the symbol $\emptyset$ denoting the unique set with no elements. That is, $\phi(\emptyset)$ is equivalent to $\phi(z) \land \forall x (x \not\in z)$.

Rewrite
\[
\exists z (\emptyset \in z \land \forall x (x \in z \rightarrow S(x) \in z))
\]
to a formula that does not contain the symbol $\emptyset$. Make sure not to reuse variables.

Exercise 2

Part 2a (challenging)
The axiom of pairing (axiom 4)
\[
\forall x \forall y \exists z (x \in z \land y \in z)
\]
says that as long as we have two sets $x$ and $y$, we have at least one set large enough that it contains both $x$ and $y$. We now express a formula $\text{PAIR}(x, y, z)$ that states that $z$ is a set containing only $x$ and $y$.

\[
\text{PAIR}(x, y, z) \leftrightarrow [x \in z \land y \in z \land \forall w (w \in z \rightarrow (w = x \lor w = y))]
\]

Use the axioms stated so far to prove that $\forall x \forall y \exists ! z \text{PAIR}(x, y, z)$.

**Hint:** For existence, you can divide the formula into two conjuncts, one equal to the axioms of pairing, and one similar to the axioms of comprehension with $\phi(w) \leftrightarrow (w = x \lor w = y)$. For uniqueness, assume there are two solutions and prove that they are equal using extensionality.

Part 2b
Using the result from part a, we are justified in defining a function $\text{pair}(x, y)$ so that
\[
\text{pair}(x, y) = z \iff \text{PAIR}(x, y, z).
\]
Instead of $\text{pair}(x, y)$, we often write \{x, y\}. If we do not want to use this new notation, we can rewrite any formula $\phi(\{x, y\})$ to $\phi(z) \land \text{PAIR}(x, y, z)$.

Write $w = \{x, \{y, z\}\}$ without the new notation (you can use the short-hand $\text{PAIR}(x, y, z)$ for the long formula).
Exercise 3

The axiom of union (axiom 5)

\[ \forall \mathcal{F} \exists A \forall Y \forall x \ [(x \in Y \land Y \in \mathcal{F}) \rightarrow x \in A], \]
says that given a set of sets, we can form the union of the inner sets. Combining this with comprehension, we can justify the notation

\[ \bigcup \mathcal{F} = \bigcup_{Y \in \mathcal{F}} Y = \{ x \mid \exists Y (x \in Y \land Y \in \mathcal{F}) \}. \]

Using the union operator, we can now build sets with more than two elements. Given sets \( a, b, c, d, e, f \), we can build the set containing exactly these as elements as follows:

\[
\bigcup \text{pair} \left( \text{pair}(a, b), \bigcup \text{pair} [\text{pair}(c, d), \text{pair}(e, f)] \right) \\
= \bigcup \text{pair} \left( \{a, b\}, \bigcup \text{pair} \{\{c, d\}, \{e, f\}\} \right) \\
= \bigcup \text{pair} \left( \{a, b\}, \bigcup \{\{c, d\}, \{e, f\}\} \right) \\
= \bigcup \{\{a, b\}, \{c, d, e, f\}\} \\
= \{a, b, c, d, e, f\}.
\]

We can build singletons \( \{x\} \) using \text{pair}(x, x). Use only \( \cup \) and \text{pair} to build the set

\( \{a, b, c, d, e, f, g\} \).

Exercise 4 (challenging)

Using union, and pairing, we define \( S(x) = x \cup \{x\} \), or

\[ S(x) = \bigcup \text{pair}(x, \text{pair}(x, x)). \]

We use this definition to create sets to represent the natural numbers.

\[
0 = \emptyset \\
1 = S(0) = \{\emptyset\} = \{0\} \\
2 = S(1) = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 = S(2) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}
\]

\[
\vdots
\]
Since for every set \( x \) we can prove that there is a set \( S(x) \), any model satisfying axioms 0, 1, 3, 4 and 5 must have an infinite domain.

The axiom of infinity (axiom 7)

\[
\exists x \[ \emptyset \in x \land \forall y(y \in x \rightarrow S(y) \in x) \]
\]

states that our domain must also contain an infinite set. That is, there is some set \( d \) where \( c_i \in d \) for an infinite number of \( c_i \).

Using comprehension with a carefully selected formula, we can form the set

\[
\omega = \{0, 1, 2, \ldots\}.
\]

That is, the set containing exactly the successor sequence starting at \( \emptyset \).

Show that \( \in \) is a strict total order on \( \omega \), that is, it behaves as the less-than relation on numbers.

A strict total order \( R \) must satisfy the following:

- \( \forall x \forall y \forall z(xRy \land yRz \rightarrow xRz) \) (transitivity)
- \( \forall x \forall y(xRy \rightarrow \neg yRx) \) (asymmetry)
- \( \forall x \forall y(xRy \lor yRx \lor x = y) \) (totality)