INF3580 – Semantic Technologies – Spring 2011

Lecture 5: Mathematical Foundations

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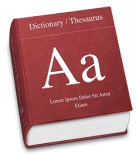
Today's Plan

Outline

Motivation

- The great thing about Semantic Technologies is. . .
- . . . Semantics!
- ~~"The study of meaning"~~
- RDF has a precisely defined semantics (=meaning)
- Mathematics is best at precise definitions
- RDF has a mathematically defined semantics



Sets: Cantor's Definition

- From the inventor of Set Theory, Georg Cantor (1845–1918):

  Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen.

- Translated:

  A "set" is any collection M of definite, distinguishable objects m of our intuition or intellect (called the "elements" of M) to be conceived as a whole.

- There are some problems with this, but it's good enough for us!

Sets

- A set is a mathematical object like a number, a function, etc.
- Knowing a set is
  - knowing what is in it
  - knowing what is not
- There is no order between elements
- Nothing can be in a set several times
- To sets A and B are equal if they contain the same elements
  - everything that is in A is also in B
  - everything that is in B is also in A

Elements, Set Equality

- Notation for finite sets:

$$\{\text{'a'}, 1, \triangle\}$$

- Contains 'a', 1, and $\triangle$, and nothing else.
- There is no order between elements

$$\{1, \triangle\} = \{\triangle, 1\}$$

- Nothing can be in a set several times

$$\{1, \triangle, \triangle\} = \{1, \triangle\}$$

- The *notation* $\{\cdots\}$ allows to write things several times!
  $\Rightarrow$ different ways of writing the same thing!
- We use $\in$ to say that something is element of a set:

$$1 \in \{\text{'a'}, 1, \triangle\}$$

$$\text{'b'} \notin \{\text{'a'}, 1, \triangle\}$$

Set Examples

- $\{3, 7, 12\}$: a set of numbers
  - $3 \in \{3, 7, 12\}$, $0 \notin \{3, 7, 12\}$
- $\{0\}$: a set with only one element
  - $0 \in \{0\}$, $1 \notin \{0\}$
- $\{\text{'a'}, \text{'b'}, \ldots, \text{'z'}\}$: a set of letters
  - $\text{'y'} \in \{\text{'a'}, \text{'b'}, \ldots, \text{'z'}\}$, $\text{'æ'} \notin \{\text{'a'}, \text{'b'}, \ldots, \text{'z'}\}$,
- The set P_{3580} of people in the lecture room right now
  - Martin $\in P_{3580}$, Albert Einstein $\notin P_{3580}$.
- $\mathbb{N} = \{1, 2, 3, \ldots\}$: the set of all natural numbers
  - $3580 \in \mathbb{N}$, $\pi \notin \mathbb{N}$.
- $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$: the set of all prime numbers
  - $257 \in \mathbb{P}$, $91 \notin \mathbb{P}$.

Know Your Elements!

- Sets with different elements are different:

$$\{1,2\} \neq \{2,3\}$$

- What about

$$\{a,b\} \text{ and } \{b,c\}?$$

- If a, b, c are *variables*, maybe

$$a = 1, \quad b = 2, \quad c = 1$$

- Then

$$\{a,b\} = \{1,2\} = \{2,1\} = \{b,c\}$$

- $\{1,2,3\}$ has 3 elements, what about $\{a,b,c\}$?

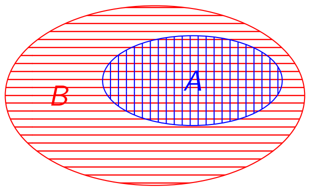
Sets as Properties

- Sets are used a lot in mathematical notation
- Often, just as a short way of writing things
- More specifically, that something has a property
- E.g. "*n* is a prime number."
- In mathematics: $n \in \mathbb{P}$
- E.g. "Martin is a human being".
- In mathematics, $m \in H$, where
  - H is the set of all human beings
  - m is Martin
- One *could* define $Prime(n)$, $Human(m)$, etc. but that is not usual
- Instead of writing "x has property XYZ" or "$XYZ(x)$",
  - let P be the set of all objects with property XYZ
  - write $x \in P$.

The Empty Set

- Sometimes, you need a set that has no elements.
- This is called the *empty set*
- Notation: $\emptyset$ or $\{\}$
- $x \notin \emptyset$, whatever x is!

Subsets



- Let A and B be sets
- *if* every element of A is also in B
- *then* A is called a *subset* of B
- This is written

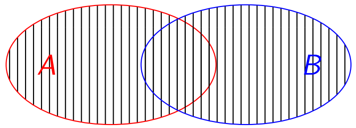
$$A \subseteq B$$

- Examples
  - $\{1\} \subseteq \{1, \text{'a'}, \triangle\}$
  - $\{1,3\} \not\subseteq \{1,2\}$
  - $\mathbb{P} \subseteq \mathbb{N}$
  - $\emptyset \subseteq A$ for any set A
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

Set Union

- The *union* of A and B contains
  - all elements of A
  - all elements of B
  - also those in both A and B
  - and nothing more.
- It is written

$$A \cup B$$



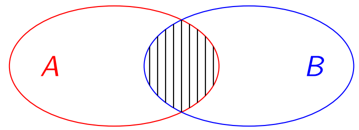
$A \cup B$

- (A cup which you pour everything into)
- Examples
  - $\{1,2\} \cup \{2,3\} = \{1,2,3\}$
  - $\{1,3,5,7,9,\ldots\} \cup \{2,4,6,8,10,\ldots\} = \mathbb{N}$
  - $\emptyset \cup \{1,2\} = \{1,2\}$

Set Intersection

- The *intersection* of A and B contains
  - those elements of A
  - that are also in B
  - and nothing more.
- It is written

$$A \cap B$$



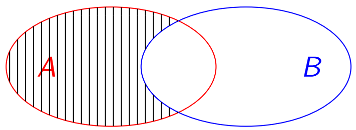
$A \cap B$

- Examples
  - $\{1,2\} \cap \{2,3\} = \{2\}$
  - $\mathbb{P} \cap \{2,4,6,8,10,\ldots\} = \{2\}$
  - $\emptyset \cap \{1,2\} = \emptyset$

Set Difference

- The *set difference* of A and B contains
  - those elements of A
  - that are *not* in B
  - and nothing more.
- It is written

$$A \setminus B$$



$A \setminus B$

- Examples
  - $\{1,2\} \setminus \{2,3\} = \{1\}$
  - $\mathbb{N} \setminus \mathbb{P} = \{1,4,6,8,9,10,12,\ldots\}$
  - $\emptyset \setminus \{1,2\} = \emptyset$
  - $\{1,2\} \setminus \emptyset = \{1,2\}$

Set Comprehensions

- Sometimes enumerating all elements is not good enough
- E.g. there are infinitely many, and "..." is too vague
- Special notation:

$$\{x \in A \mid x \text{ has some property}\}$$

- The set of those elements of A which have the property.
- Examples:
  - $\{n \in \mathbb{N} \mid n = 2k \text{ for some } k\}$: the even numbers
  - $\{n \in \mathbb{N} \mid n < 5\} = \{1,2,3,4\}$
  - $\{x \in A \mid x \notin B\} = A \setminus B$

Outline

Motivation

- RDF is all about
  - Resources (objects)
  - Their properties (`rdf:type`)
  - Their relations amongst each other
- Sets are good to group objects with some properties!
- How do we talk about relations between objects?

Pairs

- A pair is an *ordered* collection of two objects
- Written
$$\langle x, y \rangle$$
- Equal if components are equal:
$$\langle a, b \rangle = \langle x, y \rangle \quad \text{if and only if} \quad a = x \quad \text{and} \quad b = y$$
- Order matters:
$$\langle 1, \text{‘a’} \rangle \neq \langle \text{‘a’}, 1 \rangle$$
- An object can be twice in a pair:
$$\langle 1, 1 \rangle$$
- $\langle x, y \rangle$ is a pair, no matter if $x = y$ or not.

The Cross Product

- Let A and B be sets.
- Construct the set of all pairs $\langle a, b \rangle$ with $a \in A$ and $b \in B$.
- This is called the *cross product* of A and B, written
$$A \times B$$
- Example:
  - $A = \{1, 2, 3\}$, $B = \{\text{‘a’}, \text{‘b’}\}$.
  - $A \times B = \{ \ \langle 1, \text{‘a’} \rangle, \ \langle 2, \text{‘a’} \rangle, \ \langle 3, \text{‘a’} \rangle, \ \langle 1, \text{‘b’} \rangle, \ \langle 2, \text{‘b’} \rangle, \ \langle 3, \text{‘b’} \rangle \ \}$
- Why bother?
- Instead of “$\langle a, b \rangle$ is a pair of a natural number and a person in this room”...
- ... $\langle a, b \rangle \in \mathbb{N} \times P_{3580}$
- But most of all, there are subsets of cross products...

Relations

- A *relation* R between two sets A and B is...
- ...a set of pairs $\langle a, b\rangle \in A \times B$

$$R \subseteq A \times B$$

- We often write aRb to say that $\langle a, b\rangle \in R$
- Example:
  - Let $L = \{\text{‘a’}, \text{‘b’}, \ldots, \text{‘z’}\}$
  - Let $\triangleright$ relate each number between 1 and 26 to the corresponding letter in the alphabet:

$$1 \triangleright \text{'a'} \qquad 2 \triangleright \text{'b'} \qquad \ldots \qquad 26 \triangleright \text{'z'}$$

  - Then $\triangleright \subseteq \mathbb{N} \times L$:

$$\triangleright = \{\langle 1, \text{‘a’}\rangle, \langle 2, \text{‘b’}\rangle, \ldots, \langle 26, \text{‘z’}\rangle\}$$

  - And we can write:

$$\langle 1, \text{‘a’}\rangle \in \triangleright \qquad \langle 2, \text{‘b’}\rangle \in \triangleright \qquad \ldots \qquad \langle 26, \text{‘z’}\rangle \in \triangleright$$

More Relations

- A relation R *on* some set A is a relation from A to A:

$$R \subseteq A \times A = A^2$$

- Example: $<$
  - Consider the $<$ order on natural numbers:

$$1 < 2 \qquad 1 < 3 \qquad 1 < 4 \qquad \ldots \qquad 2 < 3 \qquad 2 < 4 \qquad \ldots$$

  - $< \subseteq \mathbb{N} \times \mathbb{N}$:

$$\begin{array}{lllllll} < \; = \; \{ & \langle 1,2\rangle & \langle 1,3\rangle & \langle 1,4\rangle & \ldots & \\ & & \langle 2,3\rangle & \langle 3,4\rangle & \ldots & \\ & & & \langle 3,4\rangle & \ldots & \\ & & \ldots & & & \} \end{array}$$

  - $< = \{\langle x, y\rangle \in \mathbb{N}^2 \mid x < y\}$

Family Relations

- Consider the set $S = \{\text{Homer}, \text{Marge}, \text{Bart}, \text{Lisa}, \text{Maggie}\}$.
- Define a relation P on S such that

$$x\,P\,y \quad \text{iff} \quad x \text{ is parent of } y$$

- For instance:

$$\text{Homer } P \text{ Bart} \qquad \text{Marge } P \text{ Maggie}$$

- As a set of pairs:

$$\begin{array}{llll} P = \{ & \langle\text{Homer}, \text{Bart}\rangle, & \langle\text{Homer}, \text{Lisa}\rangle, & \langle\text{Homer}, \text{Maggie}\rangle, \\ & \langle\text{Marge}, \text{Bart}\rangle, & \langle\text{Marge}, \text{Lisa}\rangle, & \langle\text{Marge}, \text{Maggie}\rangle \quad \} \subseteq S^2 \end{array}$$

- For instance:

$$\langle\text{Homer}, \text{Bart}\rangle \in P \qquad \langle\text{Marge}, \text{Maggie}\rangle \in P$$

Special Kinds of Relations

- Certain properties of relations occur in many applications
- Therefore, they are given names
- $R \subseteq A^2$ is *reflexive*

  - $x\,R\,x$ for all $x \in A$.
  - E.g. "$=$", "$\leq$" in mathematics, "has same color as", etc.
- $R \subseteq A^2$ is *symmetric*

  - If $x\,R\,y$ then $y\,R\,x$.
  - E.g. "$=$" in mathematics, friendship in facebook, etc.
- $R \subseteq A^2$ is *transitive*

  - If $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$
  - E.g. "$=$", "$\leq$", "$<$" in mathematics, "is ancestor of", etc.

Outline

Many Kinds of Logic

- In mathematical logic, many kinds of logic are considered
  - propositional logic (and, or, not)
  - description logic (a mother is a person who is female and has a child)
  - modal logic (Alice knows that Bob didn't know yesterday that...)
  - first-order logic (For all..., for some...)
- All of them formalizing different aspects of reasoning
- All of them defined mathematically
  - Syntax ($\approx$ grammar. What is a formula?)
  - Semantics (What is the meaning?)
    - proof theory: what is legal reasoning?
    - model semantics: declarative using set theory.
- For semantic technologies, description logic (DL) is most interesting
  - talks about sets and relations
- Basic concepts can be explained using predicate logic

Propositional Logic: Formulas

- Formulas are defined "by induction" or "recursively":

1 Any letter p, q, r,... is a formula

2 *if* A and B are formulas, *then*
  - $(A \land B)$ is also a formula (read: "A and B")
  - $(A \lor B)$ is also a formula (read: "A or B")
  - $\neg A$ is also a formula (read: "not A")

- Nothing else is. Only what rules [1] and [2] say is a formula.
- Examples for formulae:

$$p \quad (p \land \neg r) \quad (q \land q) \quad (q \land \neg q) \quad ((p \lor \neg q) \land (\neg p \land q))$$

- Examples for non-formulas:

$$pqr \quad p\neg q \quad \land(p$$

Propositional Formulas, Using Sets

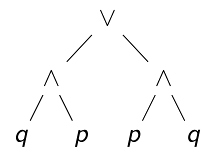
- Definition using sets:
- The set of all formulas Φ is the least set such that

1 All letters $p, q, r, \ldots \in \Phi$

2 *if* $A, B \in \Phi$, then
  - $(A \land B) \in \Phi$
  - $(A \lor B) \in \Phi$
  - $\neg A \in \Phi$

- Formulas are just a kind of strings until now:
  - no meaning
  - but every formula can be "parsed" uniquely.

$$((q \land p) \lor (p \land q))$$

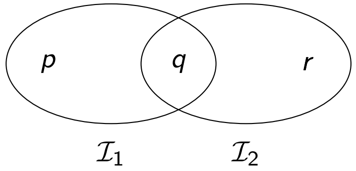


Truth

- Logic is about things being true or false, right?
- Is $(p \wedge q)$ true?
- That depends on whether p and q are true!
- *If* p is true, and q is true, then $p \wedge q$ is true
- *Otherwise*, $(p \wedge q)$ is false.
- So truth of a formula depends on the truth of the letters
- We also say the “interpretation” of the letters
- In other words, in general, truth depends on the context
- Let’s formalize this context, a.k.a. interpretation

Interpretations

- Idea: put all letters that are “true” into a set!
- Define: An *interpretation* $\mathcal{I}$ is a set of letters.
- Letter p is true in interpretation $\mathcal{I}$ if $p \in \mathcal{I}$.
- E.g., in $\mathcal{I}_1 = \{p, q\}$, p is true, but r is false.



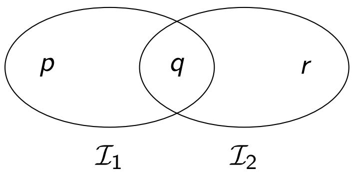
- But in $\mathcal{I}_2 = \{q, r\}$, p is false, but r is true.

Semantic Validity $\models$

- To say that p is true in $\mathcal{I}$, write

$$\mathcal{I} \models p$$

- For instance



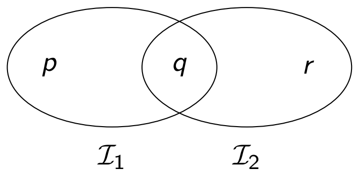
$$\mathcal{I}_1 \models p \qquad \mathcal{I}_2 \not\models p$$

- In other words, for all letters p:

$$\mathcal{I} \models p \quad \text{if and only if} \quad p \in \mathcal{I}$$

Validity of Compound Formulas

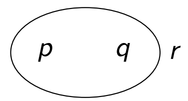
- So, is $(p \wedge q)$ true?
- That depends on whether p and q are true!
- And that depends on the interpretation.
- All right then, *given some* $\mathcal{I}$, is $(p \wedge q)$ true?
- Yes, if $\mathcal{I} \models p$ *and* $\mathcal{I} \models q$
- No, otherwise
- For instance



$$\mathcal{I}_1 \models p \wedge q \qquad \mathcal{I}_2 \not\models p \wedge q$$

Validity of Compound Formulas, cont.

- That was easy, p and q are only letters...
- ...so, is $((q \land r) \land (p \land q))$ true in $\mathcal{I}$?
- Idea: apply our rule recursively
- For any formulas A and B,...
- ...and any interpretation $\mathcal{I}$,...
- ...$\mathcal{I} \models A \land B$ if and only if $\mathcal{I} \models A$ and $\mathcal{I} \models B$
- For instance



$\mathcal{I}_1$

$$\mathcal{I}_1 \not\models ((q \land r) \land (p \land q))$$

$$\mathcal{I}_1 \not\models (q \land r) \qquad \mathcal{I}_1 \models (p \land q)$$

$$\mathcal{I}_1 \models q \quad \mathcal{I}_1 \not\models r \quad \mathcal{I}_1 \models p \quad \mathcal{I}_1 \models q$$

Semantics for $\neg$ and $\lor$

- The complete definition of $\models$ is as follows:
- For any interpretation $\mathcal{I}$, letter p, formulas A, B:
  - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$
  - $\mathcal{I} \models \neg A$ iff $\mathcal{I} \not\models A$
  - $\mathcal{I} \models (A \land B)$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
  - $\mathcal{I} \models (A \lor B)$ iff $\mathcal{I} \models A$ or $\mathcal{I} \models B$ (or both)
- Semantics of $\neg$, $\land$, $\lor$ often given as *truth table*:

A	B	$\neg A$	$A \land B$	$A \lor B$
f	f	t	f	f
f	t	t	f	t
t	f	f	f	t
t	t	f	t	t

Some Formulas Are Truer Than Others

- Is $(p \lor \neg p)$ true?
- Only two interesting interpretations:

$$\mathcal{I}_1 = \emptyset \qquad\qquad \mathcal{I}_2 = \{p\}$$

- Recursive Evaluation:

$$\mathcal{I}_1 \models (p \lor \neg p) \qquad\qquad \mathcal{I}_2 \models (p \lor \neg p)$$

$$\mathcal{I}_1 \not\models p \quad \mathcal{I}_1 \models \neg p \qquad\qquad \mathcal{I}_2 \models p \quad \mathcal{I}_2 \not\models \neg p$$

$$\mathcal{I}_1 \not\models p \qquad\qquad \mathcal{I}_2 \models p$$

- $(p \lor \neg p)$ is true in *all* interpretations!

Tautologies

- A formula A that is true in *all* interpretations is called a *tautology*
- also *logically valid*
- also a *theorem* (of propositional logic)
- written:

$$\models A$$

- $(p \lor \neg p)$ is a tautology
- True whatever p means:
  - The sky is blue or the sky is not blue.
  - Petter N. will win the race or Peter N. will not win the race.
  - The slithy toves gyre or the slithy toves do not gyre.
- Possible to derive true statements mechanically...
- ...without understanding their meaning!

Checking Tautologies

- Checking whether $\models A$ is the task of SAT-solving
- (co-)NP-complete in general (i.e. in practice exponential time)
- Small instances can be checked with a truth table:

$$\models (\neg p \lor (\neg q \lor (p \land q))) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \land q)$	$(\neg q \lor (p \land q))$	$(\neg p \lor (\neg q \lor (p \land q)))$
f	f	t	t	f	t	t
f	t	t	f	f	f	t
t	f	f	t	f	t	t
t	t	f	f	t	t	t

- Therefore: $(\neg p \lor (\neg q \lor (p \land q)))$ is a tautology!

Entailment

- Tautologies are true in all interpretations
- Some Formulas are true only under certain assumptions
- A *entails* B, written $A \models B$ if
  $\mathcal{I} \models B$
  for all interpretations $\mathcal{I}$ with $\mathcal{I} \models A$
- Also: "B is a logical consequence of A"
- Whenever A holds, also B holds
- For instance:

$$p \land q \models p$$

- Independent of meaning of p and q:
  - If it rains and the sky is blue, then it rains
  - If P.N. wins the race and the world ends, then P.N. wins the race
  - It 'tis brillig and the slythy toves do gyre, then 'tis brillig

Checking Entailment

- SAT solvers can be used to check entailment:

$$A \models B \qquad \text{if and only if} \qquad \models (\neg A \lor B)$$

- We can check simple cases with a truth table:

$$(p \land \neg q) \models \neg(\neg p \lor q) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \land \neg q)$	$(\neg p \lor q)$	$\neg(\neg p \lor q)$
f	f	t	t	f	t	f
f	t	t	f	f	t	f
t	f	f	t	t	f	t
t	t	f	f	f	t	f

- So $(p \land \neg q) \models \neg(\neg p \lor q)$
- And $\neg(\neg p \lor q) \models (p \land \neg q)$

Recap

- Sets
  - are collections of objects without order or multiplicity
  - often used to gather objects which have some property
  - can be combined using $\cap, \cup, \setminus$
- Relations
  - are sets of pairs (subset of cross product $A \times B$)
  - $x \, R \, y$ is the same as $\langle x, y \rangle \in R$
  - can be (any combination of) symmetric, reflexive, transitive
- Predicate Logic
  - has formulas built from letters, $\land$, $\lor$, $\neg$ *(syntax)*
  - which can be evaluated in an *interpretation (semantics)*
  - interpretations are sets of letters
  - recursive definition for semantics of $\land, \lor, \neg$
  - $\models A$ if $\mathcal{I} \models A$ for all $\mathcal{I}$ *(tautology)*
  - $A \models B$ if $\mathcal{I} \models B$ for all $\mathcal{I}$ with $\mathcal{I} \models A$ *(entailment)*
  - truth tables can be used for checking