Analogous findings were accumulating in the mathematics education literature. In the early 1980s, Silver (1982), Silver, Branca, and Adams (1980), and Garofalo and Lester (1985) pointed out the usefulness of the construct for mathematics educators; Lesh (1983, 1985) focused on the instability of students' conceptualizations of problems and problem situations and of the consequences of such difficulties. Speaking loosely, all of these studies dealt with the same set of issues regarding effective and resourceful problem-solving behavior. Their results can be summed up as follows: It's not just what you know; it's how, when, and whether you use it. The focus here is on two sets of studies designed to help students develop self-regulatory skills during mathematical problem solving. The studies were chosen for discussion because of (1) the explicit focus on self-regulation in both studies, (2) the amount of time each devoted to helping students develop such skills, and (3) the detailed reflections on success and failure in each.

Schoenfeld's (1985a, 1987a) problem-solving courses at the college level have as one of their major goals the development of executive or control skills. Here is a brief summary, adapted from Schoenfeld (1989d.)

The major issues are illustrated in Figures 15.3 and 15.4. Figure 15.3 shows the graph of a problem-solving attempt by a pair of students working as a team. The students read the problem, quickly chose an approach to it, and pursued that

Alan H. Schoenfeld
Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics.
In
Douglas A. Grouws (ed.)
Handbook of Research on Mathematics Teaching and Learning. A project of the National Council of Teachers of Mathematics.
Macmillan Publishing Company, New York
1992
p.355-358
The mathematician spent more than half of his allotted time trying to make sense of the problem. Rather than committing himself to any one particular direction, he did a significant amount of analyzing and (structured) exploring—not spending time in unstructured exploration or moving into implementation until he was sure he was working in the right direction. Second, note that each of the small inverted triangles in Figure 15.4 represents an explicit comment on the state of his problem solution, for example, “Hmm. I don’t know exactly where to start here” (followed by two minutes of analyzing the problem) or “OK. All I need to be able to do is [a particular technique] and I’m done” (followed by the straightforward implementation of his problem solution). It is interesting that when this faculty member began working the problem, he had fewer of the facts and procedures required to solve the problem readily accessible to him than did most of the students who were recorded working the problem. And, as he worked through the problem, the mathematician generated enough potential wild goose chases to keep an army of problem solvers busy. But he didn’t get deflected by them. By monitoring his solution with care—pursuing interesting leads and abandoning paths that didn’t seem to bear fruit—he managed to solve the problem, while the vast majority of students did not.

The general claim is that these two illustrations are relatively typical of adult student and “expert” behavior on unfamiliar problems. For the most part, students are unaware of or fail to use the executive skills demonstrated by the expert. However, it is the case that such skills can be learned as a result of explicit instruction that focuses on metacognitive aspects of mathematical thinking. That instruction takes the form of “coaching,” with active interventions as students work on problems.

Roughly one third of the time in Schoenfeld’s problem-solving classes is spent with the students working problems in small groups. The class divides into groups of three or four students and works on problems that have been distributed, while the instructor circulates through the room as “roving consultant.” As he moves through the room, he reserves the right to ask the following three questions at any time:

What (exactly) are you doing? (Can you describe it precisely?)
Why are you doing it? (How does it fit into the solution?)
How does it help you? (What will you do with the outcome when you obtain it?)

He begins asking these questions early in the term. When he does so, the students are generally at a loss regarding how to answer them. With the recognition that, despite their discomfort, he is going to continue asking those questions, the students begin to defend themselves against them by discussing the answers to them in advance. By the end of the term, this behavior has become habitual. (Note, however, that the better part of a semester is necessary to obtain such changes.)

The results of these interventions are best illustrated in Figure 15.5, which summarizes a pair of students’ problem-solving attempt after taking the course. After reading the problem, they jumped into one solution attempt which, unfortunately, was based on an unfounded assumption. They realized this a few minutes later and decided to try something else. That choice too was a bad one, and they got involved in complicated computations that kept them occupied for 8 1/2 minutes. But at that point they stopped once again. One of the students said, “No, we aren’t getting anything here. [What we’re doing isn’t justi—
In this, their solution was also typical of post-instruction attempts by the students. In contrast to the 60% of the "jump they did find" in this, the students' behavior were to have the teacher (1) serve as external monitor during search and intervention study at the middle school-level, would have had the opportunity to pursue the correct solution advanced" seventh-grade mathematics class, was to foster students' metacognitive development. Ways of achieving this goal included both "routine" and "nonroutine" problems. An example of a routine problem designed to give students experience in translating verbal statements into mathematical expressions was as follows.

Laura and Beth started reading the same book on Monday. Laura read 19 pages a day and Beth read 4 pages a day. What page was Beth on when Laura was on page 133?

The nonroutine problems used in the study included "process problems" (problems for which there is no standard algorithm for extracting or representing the given information) and problems with either superfluous or insufficient information. The instruction focused on problems amenable to particular strategies (guess-and-check, work backwards, look for patterns) and included games for whole-group activities. Assessment data and tools employed before, during, and after the instruction included written tests, clinical interviews, observations of individual and pair problem-solving sessions, and videotapes of the classroom instruction. Some of the main conclusions drawn by Lester et al. were as follows:

- There is a dynamic interaction between the mathematical concepts and processes (including metacognitive ones) used to solve problems using those concepts. That is, control processes and awareness of cognitive processes develop concurrently with an understanding of mathematical concepts.
- In order for students' problem-solving performance to improve, they must attempt to solve a variety of types of problems on a regular basis and over a prolonged period of time.
- Metacognition instruction is most effective when it takes place in a domain-specific context.
- Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.
- It is difficult for the teacher to maintain the roles of monitor, facilitator, and model in the face of classroom reality, especially when the students are having trouble with basic subject matter.
- Classroom dynamics regarding small-group activities are not as well understood as one would like, and facile assumptions that "small-group interactions are best" may not be warranted. The issue of "ideal" class configurations for problem-solving lessons needs more thought and experimentation.
- Assessment practices must reward and encourage the kinds of behaviors we wish students to demonstrate (1989, pp. 88-95).

Briefly, the findings discussed in this section are that developing self-regulatory skills in complex subject-matter domains is difficult and often involves behavior modification—"unlearning" inappropriate control behaviors developed through prior instruction. Such change can be catalyzed, but it requires a long period of time, with sustained attention to both cognitive and metacognitive processes. The task of creating the "right" instructional context, and providing the appropriate kinds of modeling and guidance, is challenging and subtle for


TABLE 15-2. Teaching Actions for Problem-Solving

<table>
<thead>
<tr>
<th>Teaching Action</th>
<th>Purpose</th>
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<tbody>
<tr>
<td><strong>BEFORE</strong></td>
<td></td>
</tr>
<tr>
<td>1. Read the problem—discuss words or phrases students may not understand</td>
<td>Illustrate the importance of reading carefully; focus on special vocabulary</td>
</tr>
<tr>
<td>2. Use whole-class discussion to focus on importance of understanding the problem</td>
<td>Focus on important data, clarification process</td>
</tr>
<tr>
<td>3. (Optional) Whole-class discussion of possible strategies to solve a problem</td>
<td>Elicit ideas for possible ways to solve the problem</td>
</tr>
<tr>
<td><strong>DURING</strong></td>
<td></td>
</tr>
<tr>
<td>4. Observe and question students to determine where they are</td>
<td>Diagnose strengths and weaknesses</td>
</tr>
<tr>
<td>5. Provide hints as needed</td>
<td>Help students past blockages</td>
</tr>
<tr>
<td>6. Provide problem extensions as needed</td>
<td>Challenge early finishers to generalize</td>
</tr>
<tr>
<td>7. Require students who obtain a solution to “answer the question”</td>
<td>Require students to look over their work and make sure it makes sense</td>
</tr>
<tr>
<td><strong>AFTER</strong></td>
<td></td>
</tr>
<tr>
<td>8. Show and discuss solutions</td>
<td>Show and name different strategies</td>
</tr>
<tr>
<td>9. Relate to previously solved problems or have students solve extensions</td>
<td>Demonstrate general applicability of problem solving strategies</td>
</tr>
<tr>
<td>10. Discuss special features, e.g. pictures</td>
<td>Show how features may influence approach</td>
</tr>
</tbody>
</table>

(Adapted from Lester et al., 1989, p. 26)

Beliefs and Affects

Once upon a time there was a sharply delineated distinction between the cognitive and affective domains, as reflected in the two volumes of Bloom's (1956) *Taxonomy of Educational Objectives*. Concepts such as mathematics anxiety, for example, clearly resided in the affective domain and were measured by questionnaires dealing with how the individual feels about mathematics (see, for example, Suinn, Edie, Nicoletti, & Spinelli, 1972). Concepts such as mathematics achievement and problem solving resided within the cognitive domain and were assessed by tests focusing on subject-matter knowledge alone. As our vision gets clearer, however, the boundaries between those two domains become increasingly blurred.

Given the space constraints, to review the relevant literature or even try to give a sense of it would be an impossibility. Fortunately, one can point to McLeod, Chapter 23, this volume, and to books such as McLeod and Adams's (1989) *Affect and Mathematical Problem Solving: A New Perspective* as authoritative starting points for a discussion of affect. Beliefs—to be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior—will receive a telegraphic discussion. The discussion will take place in three parts: student beliefs, teacher beliefs, and general societal beliefs about doing mathematics. There is a fairly extensive literature on the first, a moderate but growing literature on the second, and a small literature on the third. Hence, length of discussion does not correlate with the size of the literature base.

**Student Beliefs.** As an introduction to the topic, we recall Lampert's commentary: