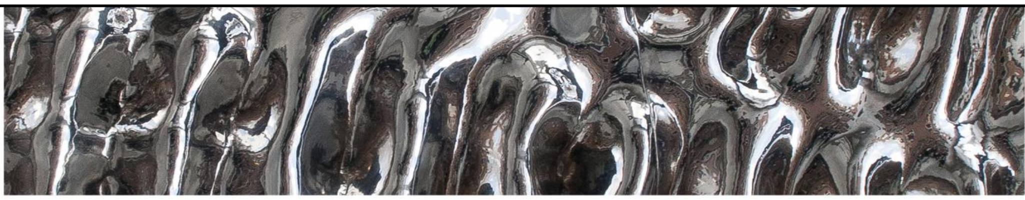

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INF4420

Discrete time signals

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Spring 2013

Outline

- Impulse sampling
- *z*-Transform
- Frequency response
- Stability

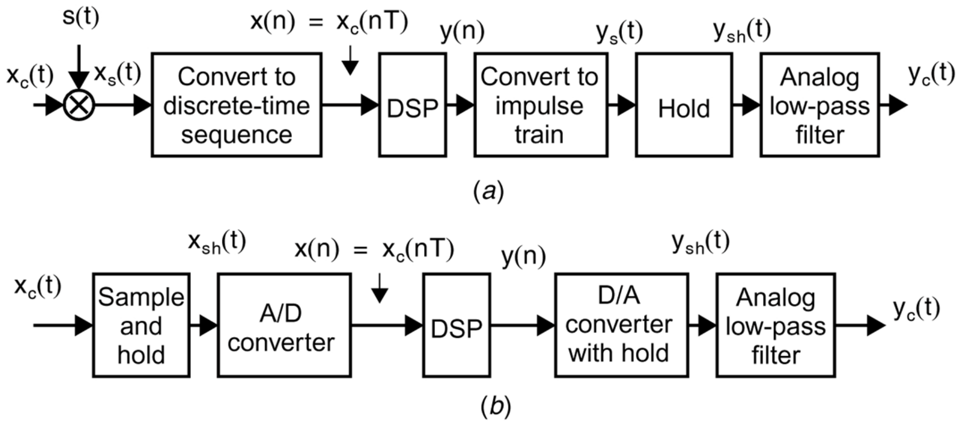
Introduction

- More practical to do processing on sampled signals in many cases
- Sampled + quantized signals = digital
- Inputs and outputs are not sampled
- How does sampling affect the signals?
- Tools for analyzing sampled signals and systems ("discrete Laplace transform", the *z*-transform)

Introduction

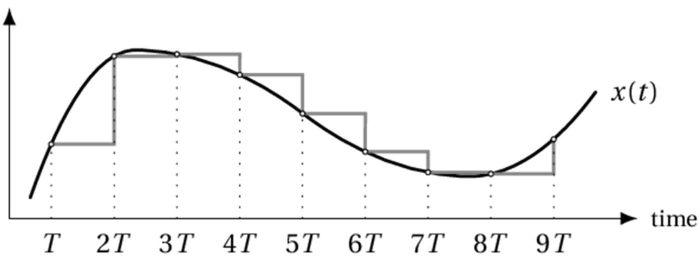
- We have already seen sample and hold circuits
- We can also realize integrators, filters, etc. as sampled analog systems—switched capacitor techniques. Discrete time, continuous amplitude.
- Digital processing is efficient and robust, usually preferred where applicable. Sampling also applies to digital.

Introduction

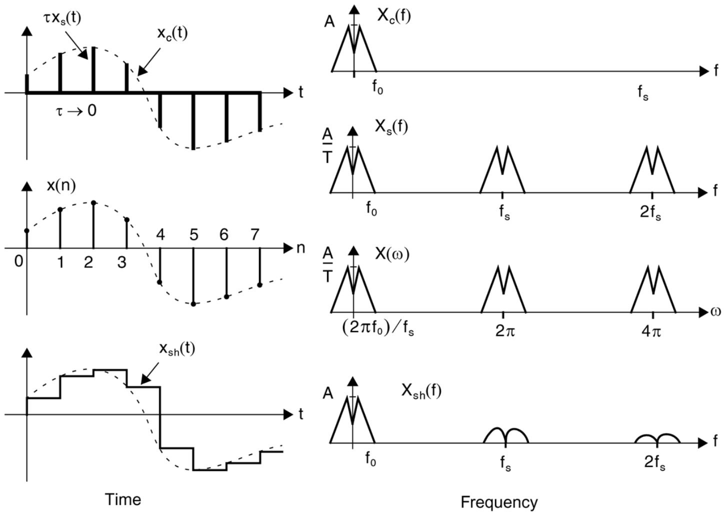


Introduction

- Sample a continuous time input signal at uniformely spaced time points.
- Output is a discrete sequence of values (in theory).



Introduction



Sampling

Laplace transform:

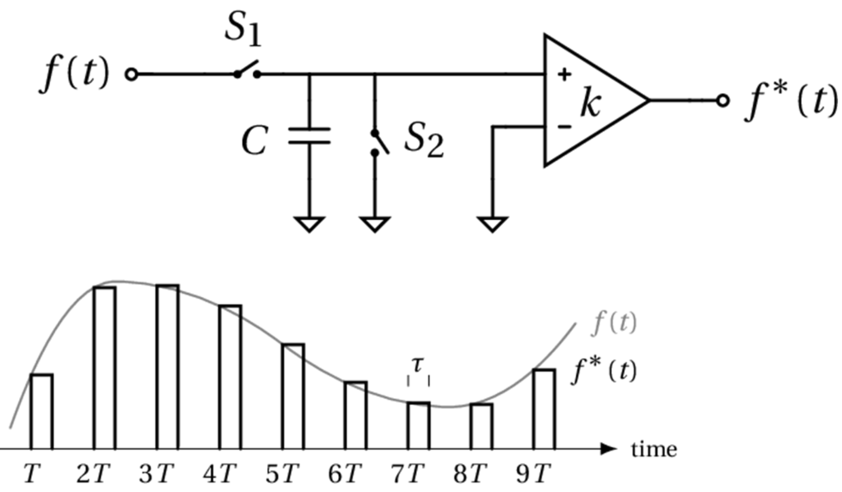
Input signal

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Fourier transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Sampling



Sampling

Impulse sampling

- Choose τ infinitely narrow
- Choose the gain $k = \frac{1}{\tau}$.

$\rightarrow$ The area of the pulse at nT is equal to the instantaneous value of the input at nT, $f(nT)$.

The signal is still defined for all time, so we can use the Laplace transform for analysis.

Sampling

- Modelling the sampled output, $f^*(t)$
- We will model the sampled output in the time domain
- Then find an equivalent representation in the Laplace domain
- We will model each pulse independently and the whole signal by summing all pulses

Sampling

Modeling a single pulse using step functions

Step function:

$$u(t) \equiv \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$$

Single pulse:

$$f(nT) \cdot \big(u(t - nT) - u(t - nT - \tau)\big)$$

Sampling

Sum all the pulses to get the sampled signal:

$$f^*(t) = k\sum_{n=0}^{\infty} f(nT)\cdot\big(u(t-nT)-u(t-nT-\tau)\big)$$

Sampling

Transforming the time domain model to the Laplace domain

Relevant Laplace transforms

$$\begin{array}{ccc} f(t) & \leftrightarrow & F(s) \\ u(t) & \leftrightarrow & s^{-1} \\ f(t-a)u(t-a), a \geq 0 & \leftrightarrow & e^{-as}F(s) \end{array}$$

Sampling

Time domain model from before

$$f^*(t) = k \sum_{n=0}^{\infty} f(nT) \cdot \big(u(t-nT) - u(t-nT-\tau)\big)$$

Laplace domain

$$F^*(s) = k \sum_{n=0}^{\infty} f(nT) \left(\frac{e^{-snT}}{s} - \frac{e^{-s(nT+\tau)}}{s} \right)$$
$$= \frac{k(1-e^{-s\tau})}{s} \sum_{n=0}^{\infty} f(nT) e^{-snT}$$

Sampling and the z-transform

$$F^*(s) = \frac{k(1-e^{-s\tau})}{s} \sum_{n=0}^{\infty} f(nT) e^{-snT}$$

Impulse sampling: $k = \frac{1}{\tau}, \tau \to 0 \; (e^x \approx 1 + x)$

$$F^*(s) \approx \sum_{n=0}^{\infty} f(nT) \cdot e^{-snT} \equiv \sum_{n=0}^{\infty} f(nT) \cdot z^{-n}$$

The z-transform

$$z \equiv e^{sT}$$

$$X(z) \equiv \sum_{n=0}^{\infty} x(nT) \cdot z^{-n} = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

- Delay by k samples, $z^{-k} \cdot X(z)$
- Convolution in time $\leftrightarrow$ multiplication in z

Frequency response

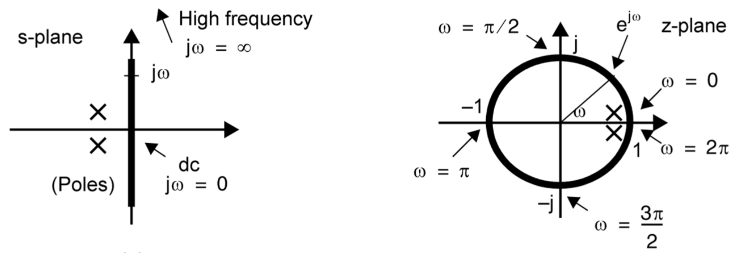
- Use the Laplace domain description of the sampled signal
- As before, substitute $s = j\omega$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(nT) \cdot e^{-j\omega nT}$$

- $X(j\omega)$ is the Fourier transform of the impulse sampled input signal, $x(t)$.
- e^{jx} is cyclic, $e^{jx} = \cos x + j \sin x$

Frequency response

- We go from the z-transform to the frequency response by substituting $z = e^{j\omega T}$
- As we sweep ω we trace out the unit circle



(a) (b)

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Frequency response

Rewriting to use frequency (Hz), rather than radian frequency

$$X(f) = \sum_{n=-\infty}^{\infty} x(nT) \cdot e^{-j2\pi f nT}$$

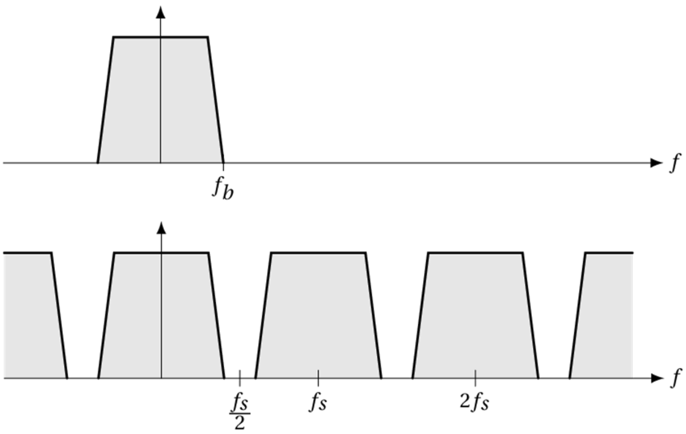
Because e^{jx} is cyclic, $f_1 = k \cdot f_s + f_1$, where f_1 is an arbitrary frequency, k is any integer and f_s is the sampling frequency (T^{-1}).

Frequency response

The frequency spectrum repeats. We can only uniquely represent frequencies from DC to $\frac{f_s}{2}$ (the Nyquist frequency).

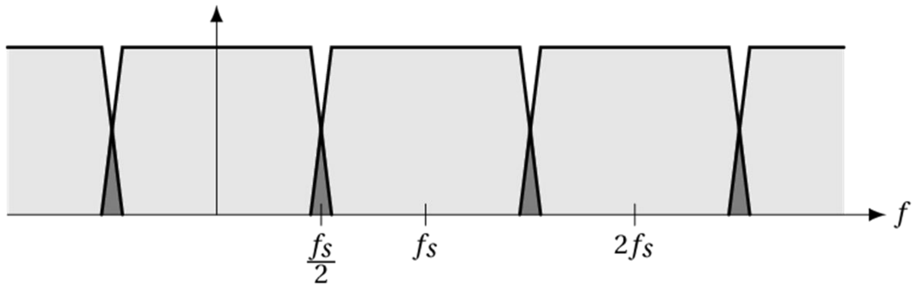
Important practical consequence: We must band limit the signal before sampling to avoid *aliasing*. A non-linear distortion.

Frequency response



Frequency response

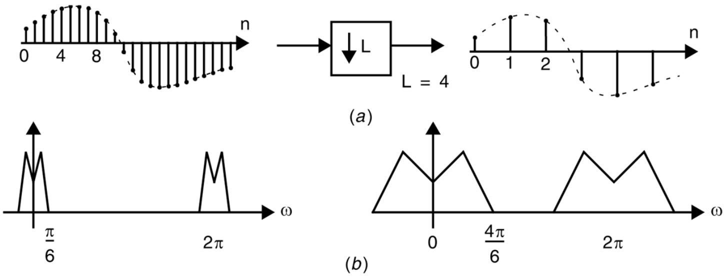
If the signal contains frequencies beyond $\frac{f_s}{2}$, sampling results in in aliasing. Images of the signal interfere.



Sampling rate conversion

- Changing the sampling rate after sampling
- We come back to this when discussing oversampled converters
- Oversampling = sampling faster than the Nyquist frequency would indicate
- Upsampling is increasing the sampling rate (number of samples per unit of time)
- Downsampling is decreasing the sampling rate

Downsampling

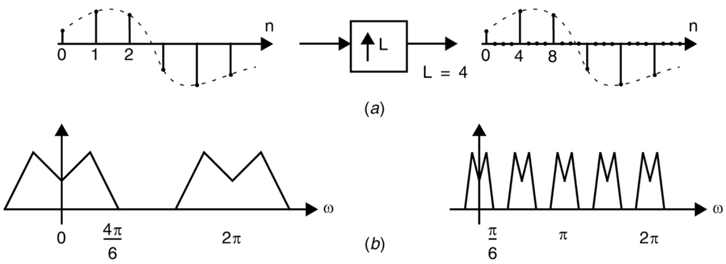


Keep every n-th sample.

Downsample too much: Aliasing

Upsampling

Insert *n* zero valued samples between each original sample, and low-pass filter. Requires gain to maintain the signal level.



Discrete time filters

Analog filters use integrators, s^{-1}, as building blocks to implement filter functions. Discrete time filters use delay, z^{-1}.

Example:

Time domain: $y[n+1] = bx[n] + ay[n]$

z-domain: $zY(z) = bX(z) + aY(z)$

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{b}{z-a}$$

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Discrete time filters

Frequency response, $z = e^{j\omega}$ (ω normalized to the sampling frequency, really $z = e^{j\omega T}$)

$$H(e^{j\omega}) = \frac{b}{e^{j\omega} - a}$$

DC is $z = e^{j0} = 1$.

The sampling frequency is $z = e^{j2\pi} = 1$ (also).

Sufficient to evaluate the frequency response from 0 to π due to symmetry (for real signals).

Stability

$$y[n+1] = bx[n] + ay[n]$$

If $a > 1$, the output grows without bounds. Not stable.

$$H(z) = \frac{b}{z-a}$$

In a stable system, all poles are inside the unit circle

- $a = 1$: Discrete time integrator
- $a = -1$: Oscillator

IIR filters

$y[n+1] = bx[n] + ay[n]$ is an infinite impulse response filter.

Single impulse input ($x[0] = 1, 0$ otherwise) results in an output that decays towards zero, but (in theory) never reaches zero. If we try to characterize the filter by its impulse response, we need an infinite number of outputs to characterize it.

FIR filters

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

Is a FIR (finite impulse response filter).

$$H(z) = \frac{1}{3}\sum_{i=0}^{2} z^{-i}$$

FIR filters are inherently stable but require higher order (more delay elements) than IIR.

Bilinear transform

- Mapping between continuous and discrete time
- Design the filter as a continuous time transfer function and map it to the *z*-domain

$$s = \frac{z-1}{z+1}, \text{ conversely, } z = \frac{1+s}{1-s}$$

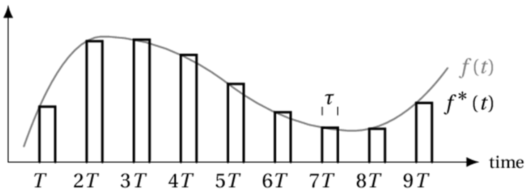
- $s = 0$ maps to $z = 1$ (DC)
- $s = \infty$ maps to $z = -1 \left(\frac{f_s}{2}\right)$
- First order approximation

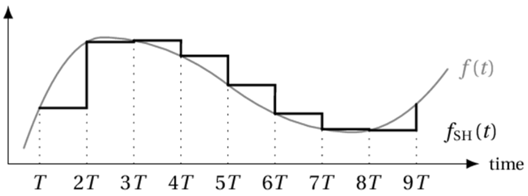
Sample and hold

We modeled impulse sampling by letting $\tau \to 0$.

For the sample and hold, we use the same model, but let $\tau \to T$.

Use this to find the transfer function of SH





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Sample and hold

$$F^*(s) = \underbrace{\frac{k(1-e^{-s\tau})}{s}}_{\text{Impulse sampling} \approx 1} \sum_{n=0}^{\infty} f(nT)e^{-snT}$$

Sample and hold:

The pulse lasts for the full sampling period, T

$$F^*(s) = \frac{k(1-e^{-sT})}{s} \sum_{n=0}^{\infty} f(nT)e^{-snT}$$

Sample and hold

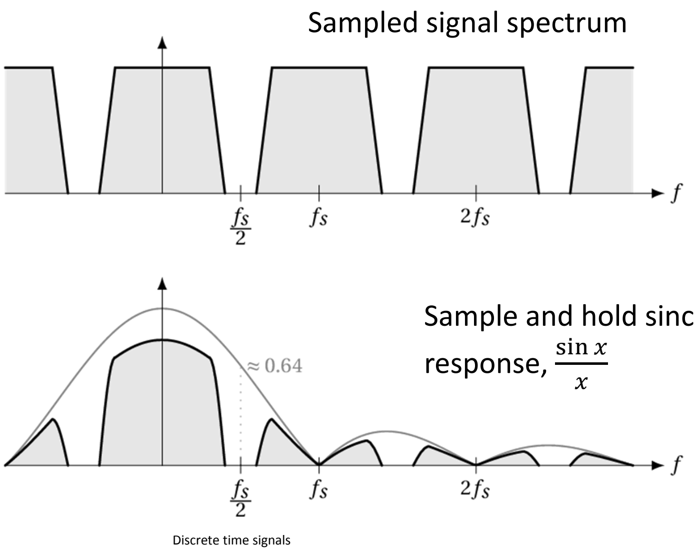
The sample and hold shapes spectrum

$$H_{SH}(s) \equiv \frac{1 - e^{-sT}}{s}$$

Frequency (magnitude) response of the SH

$$|H_{SH}(j\omega)| = T\frac{\left|\sin\frac{\omega T}{2}\right|}{\left|\frac{\omega T}{2}\right|}$$

Sample and hold



References

Gregorian and Temes, *Analog MOS Integrated Circuits for Signal Processing*, Wiley, 1986