Adaptive beamforming

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Slide 2: Chapter 7: Adaptive array processing

Minimum variance beamforming

- Generalized sidelobe canceler
- Signal coherence
- Spatial smoothing to de-correlate sources

Eigenanalysis methods

- Signal/noise subspaces
- Eigenvector method
- MUSIC

Slide 3: Delay-and-sum

\[ \text{Delay-and-sum} \]

\[ \Delta_0 \quad \Delta_1 \quad \Delta_2 \quad \Delta_3 \]

\[ \text{stacking} \triangleq \text{adjustment of } \Delta_0 \ldots \Delta_{M-1} \]

Slide 4: Delay-and-sum, continued
• $z(t) = \sum_{m=0}^{M-1} w_m e^{-j\omega \Delta_m} y_m(t)$

• $w_m$: weight on signal $m \Rightarrow$ shading = apodization

• Conventional D-A-S: $w$ independent of recorded signal data

Slide 5: Delay-and-sum on vector form

• Monochromatic source: $y_m(t) = e^{j(\omega t - \tilde{k} \cdot \tilde{x}_m)}$.

• Delayed signal: $y_m(t - \Delta_m) = y_m(t) e^{-j\omega \Delta_m}$

Define:

Weights: $w \triangleq \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix}$

Received signals: $\mathbf{Y}(t) \triangleq \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{M-1}(t) \end{bmatrix}$

Delayed $\mathbf{Y}(t)$: $\tilde{Y}(t) \triangleq \begin{bmatrix} y_0(t) e^{-j\omega \Delta_1} \\ y_1(t) e^{-j\omega \Delta_1} \\ \vdots \\ y_{M-1}(t) e^{-j\omega \Delta_{M-1}} \end{bmatrix}$

• Beamf. output: $z(t) = \sum_{m=0}^{M-1} w_m e^{-j\omega \Delta_m} y_m(t) = w^H \mathbf{Y}(t)$

• Power of $z(t)$:

$$P(z(t)) \triangleq \mathbb{E}\{|z(t)|^2\} = \mathbb{E}\{(w^H \mathbf{Y})(w^H \mathbf{Y})^H\} = \mathbb{E}\{w^H \mathbf{Y} \mathbf{Y}^H w\} = w^H \mathbb{E}\{\mathbf{Y} \mathbf{Y}^H\} w = w^H \mathbf{R} w$$

Slide 5: Delay-and-sum on vector form, continued

Spatial covariance matrix in case of steering

$$\mathbf{R} \triangleq \mathbb{E}\{\mathbf{Y} \mathbf{Y}^H\} = \begin{bmatrix} \mathbb{E}\{y_0(t) e^{-j\omega \Delta_0} y_0(t) e^{+j\omega \Delta_0}\} & \mathbb{E}\{y_0(t) e^{-j\omega \Delta_0} y_1(t) e^{+j\omega \Delta_1}\} & \cdots & \mathbb{E}\{y_0(t) e^{-j\omega \Delta_0} y_{M-1}(t) e^{+j\omega \Delta_{M-1}}\} \\ \mathbb{E}\{y_1(t) e^{-j\omega \Delta_1} y_0(t) e^{+j\omega \Delta_0}\} & \mathbb{E}\{y_1(t) e^{-j\omega \Delta_1} y_1(t) e^{+j\omega \Delta_1}\} & \cdots & \mathbb{E}\{y_1(t) e^{-j\omega \Delta_1} y_{M-1}(t) e^{+j\omega \Delta_{M-1}}\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}\{y_{M-1}(t) e^{-j\omega \Delta_{M-1}} y_0(t) e^{+j\omega \Delta_0}\} & \mathbb{E}\{y_{M-1}(t) e^{-j\omega \Delta_{M-1}} y_1(t) e^{+j\omega \Delta_1}\} & \cdots & \mathbb{E}\{y_{M-1}(t) e^{-j\omega \Delta_{M-1}} y_{M-1}(t) e^{+j\omega \Delta_{M-1}}\} \end{bmatrix}$$

Slide 6: Delay-and-sum on vector form, continued

Define:

Steering vector:

$$\mathbf{e} \triangleq \begin{bmatrix} e^{-j\tilde{k} \cdot \tilde{x}_0} \\ e^{-j\tilde{k} \cdot \tilde{x}_1} \\ \vdots \\ e^{-j\tilde{k} \cdot \tilde{x}_{M-1}} \end{bmatrix}$$

• $z(t) = w^H \mathbf{Y}$

• Power of $z(t)$:

$$P(\mathbf{e}) \triangleq \mathbb{E}\{|z(t)|^2\} (\mathbf{e}) = w^H \mathbf{R}(\mathbf{e}) w$$

if $w_m = 1, \forall m$:

$$P(\mathbf{e}) \triangleq \mathbb{E}\{|z(t)|^2\} (\mathbf{e}) = w^H \mathbb{R}(\mathbf{e}) 1$$
Slide 7: **Delay-and-sum on vector form, continued**

**About e:**

- *Steering vector*

\[
e = \begin{bmatrix}
e^{-j\vec{k} \cdot \vec{x}_0} \\
e^{-j\vec{k} \cdot \vec{x}_1} \\
\vdots \\
e^{-j\vec{k} \cdot \vec{x}_{M-1}}
\end{bmatrix}
\]

- Contains delays to focus in specific direction
- Represents unit amplitude signal, propagating in \( \vec{k} \) direction

*In conventional D-A-S: \( \mathbf{w} \) independent of received signal data*

Slide 8: **Estimation of spatial covariance matrix**

- Averaging in time
- Averaging in space = spatial smoothing = subarray averaging

Slide 9: **Minimum variance beamforming**

Assume narrow-band signals

- Adaptive method. [Latin: *adaptare* “to fit to”]
- Allow \( w_m \) to also be complex and/or negative
- Sensor weights defined *not only as function of problem geometry, but also as function of received signals*
Slide 10: Minimum variance beamforming, continued

• “Minimum variance” = “Capon” = “Maximum likelihood” (beamforming)

• Constrained optimization problem

• New steering direction $\mathbf{e} \Rightarrow$ new calculation of element weights $w_m$

\[
\begin{align*}
\mathbf{w} &\triangleq \begin{bmatrix}
w_0 \\
\vdots \\
w_{M-1}
\end{bmatrix} \Rightarrow \min_{\mathbf{w}} \left( \mathbf{w}^H \mathbf{R} \mathbf{w} \right) \\
\text{constraint: } \mathbf{w}^H \mathbf{e} = 1 \\
\Rightarrow \mathbf{w} &\Rightarrow \mathbf{P}(\mathbf{e}) = \frac{1}{\mathbf{e}^H \mathbf{R}^{-1} \mathbf{e}} \\
\end{align*}
\]

• [beampattern] $\neq$ [steered response]

Slide 11: Minimum variance beamforming, continued

Example

Uniform linear array, $M = 10$
Comparing:
• Delay-and-sum
• Minimum-variance

Investigating:
• Steered responses
• Beampatterns

Slide 12: Comparison: delay-and-sum / minimum-variance

1 source
Slide 13: **Comparison: delay-and-sum / minimum-variance**

2 sources
Slide 14: **Comparison: delay-and-sum / minimum-variance**

2 closely spaced sources
Slide 15: **Example: Ultrasound imaging**

Experimental setup

Slide 16: **Example: Ultrasound imaging, continued**

Resulting images
Slide 17: **Example: Ultrasound imaging, continued**

Experimental data, heart phantom

Slide 18: **Sound example: Microphone array**

Slide 19: **Sound example, continued**

- Single microphone `mix.wav`
- Delay-and-sum `gaute_DAS.wav`
• Minimum-variance

Slide 20: **Minimum-variance beamformer:**

Uncorrelated signals required for optimum functioning

- Correlated signals may give signal cancellation
- Although constraint is fulfilled: signal propagating in the “look-direction” may be canceled by a correlated interferer

Slide 21: **Minimum-variance / D-A-S example**

2 correlated sources

![Graph showing steered response with MV and DAS labels.](image)

Slide 22: **Generalized sidelobe canceler**

- Conventional D-A-S weights $w_c$: steering in assumed propagation direction
- Adaptive/canceler portion: removal of other signals
- Blocking matrix $B$, blocks signal from assumed propagation direction from coming into “canceler part”
- $w_a$: adaptive weights to emphasize what is to be removed
- Minimize total power. Unconstrained minimization:

$$
\min_{w_a} (w_c - B^H w_a)^H R (w_c - B^H w_a) \Rightarrow w_a = (BRB^H)^{-1} BR w_c
$$

- Full details: D&J pp. 369–371
Slide 23: **Eigenanalysis and Fourier analysis**

(Chapter 7.3.1)

Eigenvalues / Eigenvectors of $R$

$$Rv_i = \lambda_i v_i$$

- $R$ **Hermitian** (self-adjoint) $\iff$ $R = R^H$
- $R$ **Positive semidefinite** $\iff$ $x^H Rx \geq 0, \forall x \neq 0$

- Eigenvalues real & positive: $\lambda_i \geq 0$, $\forall i$
- Eigenvectors $v_i$: orthogonal

$$\Rightarrow \text{For } \mathbf{V} = \begin{bmatrix} v_1 & v_2 & \ldots & v_M \end{bmatrix}: \mathbf{V} \text{ is unitary matrix}$$

$$\iff \mathbf{V}^H \mathbf{V} = \mathbf{I} \iff \mathbf{V}^{-1} = \mathbf{V}^H$$

Slide 24: **Eigenvalue decomposition**

Eigendecomposition of $R$ & $R^{-1}$: *The spectral theorem*

- $R = \sum_{i=1}^{M} \lambda_i v_i v_i^H = \mathbf{V} \Lambda \mathbf{V}^H$
- $R^{-1} = \sum_{i=1}^{M} \frac{1}{\lambda_i} v_i v_i^H = \mathbf{V} \Lambda^{-1} \mathbf{V}^H$
- *Note above:* Same eigenvectors, but inverse eigenvalues


$$R = \mathbf{V} \Lambda \mathbf{V}^H = \underbrace{\mathbf{V}_s \Lambda_s \mathbf{V}_s^H}_{\text{from signal}} + \underbrace{\mathbf{V}_n \Lambda_n \mathbf{V}_n^H}_{\text{from noise}}$$

Slide 26: **Beamformer output powers**
Minimum variance, $\hat{R} = E\{YY^H\}$: “full $R$ estimate”

$$P_{MV}(e) = \begin{bmatrix} \frac{1}{\text{signal-plus-noise subspace}} & \frac{1}{\text{noise subspace}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Eigenvector method, $\hat{R} = \sum_{i=N_s+1}^{M} \lambda_i v_i v_i^H$: “noise-only $R$ estimate”

$$P_{EV}(e) = \begin{bmatrix} \frac{1}{\text{noise subspace}} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

MUSIC, $\hat{R} = \sum_{i=N_s+1}^{M} v_i v_i^H$: “normalized noise-only $R$ estimate”

$$P_{MUSIC}(e) = \begin{bmatrix} \frac{1}{\text{noise subspace}} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

Slide 27: [Signal + noise], & [noise] subspaces, continued

- $R$ always Hermitian $\Rightarrow$ $\forall$ eigenvectors orthogonal
- $N_s$ largest eigenvectors: span the [signal+noise subspace]
- $M - N_s$ smallest span the [noise subspace]
- Signal steering vectors $\in$ [signal+noise subspace], $\perp$ [noise subspace]
- [Largest eigenvectors] $\neq$ [signal vectors] However: signal vectors are linear combinations of [largest eigenvectors]

Slide 28: EV / MUSIC

- (Inverse of) projection of possible steering vectors onto the noise subspace
- MUSIC: Dropping $\lambda_i$ in $\hat{R} \iff$ noise whitening
- EV & MUSIC: peaks at Directions-Of-Arrival (DOAs)
  - No amplitude preservation
  - Require knowledge of # signals, $N_s$
  - How to estimate $N_s$?
Slide 29: **MV / EV / MUSIC comparison**

![Graph showing comparison between MV, EV, and MUSIC]

Slide 30: **Situation with signal coherence**

\[ R = \text{SCS}^H + K_n \]

- signal coherence ⇒ rank deficient signal matrix \( \text{SCS}^H \)
- For minimum-variance: May cause signal cancellation
- Gives signal+noise-subspace consisting of linear combinations of signal vectors
- EV & MUSIC: may fail to produce peaks at DOA locations
- Want to reduce cross-correlation terms in \( C \).
- Cure: Spatial smoothing (subarray averaging)