11.1 Taylor polynom av grad 4 til $f(x) = e^{x^2}$ i punktet 0

$f(x) = e^{x^2}$

$f'(x) = 2x\, e^{x^2}$

$f''(x) = 2e^{x^2} + 2x \cdot 2x e^{x^2} = (2+4x^2)e^{x^2}$

$f'''(x) = \cancel{(2+4x^2)e^{x^2}}\ 8x e^{x^2} + 2x(2+4x^2)e^{x^2} = (12x + 8x^3)e^{x^2}$

$f^{(4)}(x) = (12 + 24x^2)e^{x^2} + (12x + 8x^3)2x e^{x^2}$

$= (12 + 48x^2 + 24x^3)e^{x^2}$

$f(0) = e^0 = 1$

$f'(0) = 0$

$f''(0) = (2+0)e^0 = 2$

$f'''(0) = 0$

$f^{(4)}(0) = 12$

$T_4 f(x; a=0) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$

$= 1 + x^2 + \frac{1}{2}x^4$

11.1.10 Taylor-polynom av grad 3 til $f(x) = x^4 - 3x^2 + 2x - 7$ i punktet 1.

$$f(x) = x^4 - 3x^2 + 2x - 7 \qquad f(1) = -7$$

$$f'(x) = 4x^3 - 6x + 2 \qquad f'(1) = 0$$

$$f''(x) = 12x^2 - 6 \qquad f''(1) = 6$$

$$f'''(x) = 24x \qquad f'''(1) = 24$$

$$T_3(\cancel{3}x) = \frac{-7}{0!}\cancel{1} + 0 + \frac{6}{2!}(x-1)^2 + \frac{24}{4!}(x-1)^3$$

$$= -7 + 3(x-1)^2 + (x-1)^3$$

11.2.1 Grad 4 Punkt 0

$f(x) = e^x$ $\qquad f(0) = f'(0) = \dots = f^{(4)}(0) = 1$

$f'(x) = e^x$

$\vdots$

$f^{(4)}(x) = e^x$

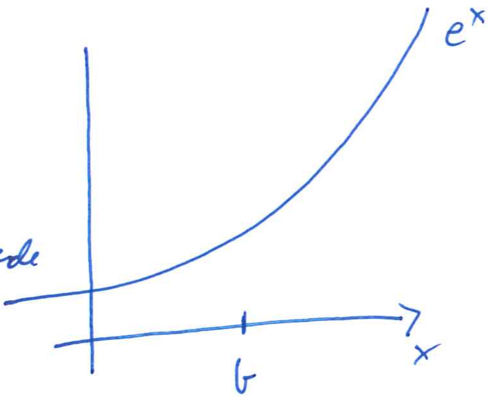
$$T_4f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

Vis at $\left|R_4f(b)\right| \leq \frac{e^b}{120} b^5 \qquad b \geq 0.$

Vi har at

$$R_4f(b) = \frac{f^{(5)}(c)}{(4+1)!}(x-0)^5 \qquad c \in (0,b)$$



Vi har

$f^{(5)}(x) = e^x$ som er strengt voksende

dvs. $e^y < e^z \Rightarrow y < z$

For $c \in (0,b)$ må derfor $\left|f^{(5)}(c)\right| < \left|f^{(5)}(b)\right|$

Det gir

$\left|R_4f(b)\right|$

11.2.2

Taylor-polynom grad 4 om ~~...~~ $a=0$ til ~~...~~ $f(x)=\sin(x)$

$f(x)=\sin x \qquad f(0)=0$

$f'(x)=\cos x \qquad f'(0)=1$

$f''(x)=-\sin x \qquad f''(0)=0$

$f'''(x)=-\cos x \qquad f'''(0)=-1$

$f^{(4)}(x)=\sin x \qquad f^{(4)}(0)=0$

$$T_4 f(x)=\frac{f(0)}{0!}+\frac{f'(0)}{1!}(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3+\frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= x-\frac{1}{6}x^3$$

Feil:

$f(x)=T_4 f(x)+R_4 f(x)$

$$R_4 f(x)=\frac{f^{(5)}(c)}{5!}(x-0)^5$$

$$|f(6)-T_4 f(6)|\leq |R_4 f(6)|$$

$f^{(5)}(x)=\cos(x)$

$$=\left|\frac{\cos(c)}{5!}6^5\right|\leq\frac{|6|^5}{5!}$$

11.2.5 Taylor-Polynom grad 2 $\qquad f(x)=\sqrt{x}$ om $a=100$

$f(x)=x^{\frac{1}{2}}$ $\qquad f(100)=(100)^{\frac{1}{2}}=10$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$ $\qquad f'(100)=\frac{1}{2}(100)^{-\frac{1}{2}}=\frac{1}{20}$

$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$ $\qquad f''(100)=-\frac{1}{4}(100)^{-\frac{3}{2}}=-\frac{1}{4}\frac{1}{(\sqrt{100})^3}=-\frac{1}{4}\frac{1}{10^3}=-\frac{1}{4000}$

$$T_2 f(x) = f(100) + f'(100)(x-100) + \frac{f''(100)(x-100)^2}{2}$$

$$= 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2$$

Finn en tilnærmet verdi til $\sqrt{101}$, anslå nøyaktigheten

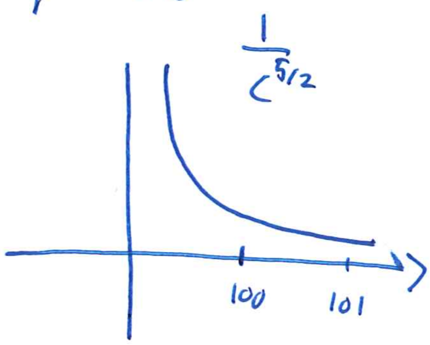
$$T_2 f(101) = 10 + \frac{1}{20} - \frac{1}{8000}$$

Finner restledd

$f'''(x)=\frac{3}{8}x^{-\frac{5}{2}}$ $\qquad c \in (100,101)$

$$\left|f(\cancel{x}^{101}) - T_2 f(\cancel{x}^{101})\right| \le \left|R_2 f(101)\right| = \left|\frac{\frac{3}{8}c^{-\frac{5}{2}}}{3!}(\cancel{x}^{101}-100)^3\right|$$

$$= \left|\frac{1}{16}\frac{1}{c^{\frac{5}{2}}}\right|$$



$$\le \left|\frac{1}{16}\frac{1}{100^{5/2}}\right|$$

$$\le \frac{1}{16\cdot 10^5} = 6{,}25\times 10^{-7}$$

11.2.6

Finn $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

Fra oppg. 11.2.1 vet vi at

$T_2 e^x = 1 + x + \frac{1}{2} x^2$ for $a = 0$

og $R_2 e^x = \dfrac{e^c}{6} x^3$ der c ligger mellom 0 og x.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{T_2 e^x + R_2 e^x - 1 - x}{x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{1} + \cancel{x} + \frac{1}{2} \cancel{x^2} + \frac{e^c x^{3}}{6} \cancel{-1} \cancel{- x}}{\cancel{x^2}}$$

$$= \lim_{x \to 0} \frac{1}{2} + \frac{e^c x}{6} = \frac{1}{2}$$

11.2.9 Fin $\int_0^1 \frac{1-e^{-t}}{t}\,dt$ meden nøyaktighet på 10^{-3}

$$T_n e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{x^n}{n!}$$

$$R_n e^x = \frac{e^{c(x)}}{(n+1)!} x^{n+1}$$

Setter $x = -t$. Da får vi

$$T_n e^{-t} = 1 + (-t) + \frac{1}{2}(-t)^2 + \cdots + \frac{(-t)^n}{n!}$$

$$= 1 - t + \frac{1}{2}(t)^2 + \cdots + (-1)^n \frac{t^n}{n!}$$

$$R_n e^{-t} = \frac{e^{c(t)}}{n+1} (-1)^{n+1} t^{n+1} \qquad c(t) \in (-t, 0).$$

Det gir

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt = \int_0^1 \frac{1-T_n e^{-t}}{t}\,dt + \int_0^1 \frac{-R_n e^{-t}}{t}\,dt$$

$$\left| \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_0^1 \frac{1-T_n e^{-t}}{t} \right| = \left| \int_0^1 \frac{-R_n e^{-t}}{t}\,dt \right|$$

Vi må finne n slik at denne er mindre enn 10^{-3}

For $t \in [0,1]$ vil $c(t) \in [-1, 0)$. Da er $|e^{c(t)}| \le 1$.

$$\left| \int_0^1 \frac{-R_n e^{-t}}{t}\,dt \right| \le \int_0^1 \frac{\left| -\overbrace{(-1)^{n+1}}^{1} \overbrace{e^{c(t)}}^{\le 1} t^{n+1} \right|}{t\,(n+1)!}\,dt$$

$$\le \int_0^1 \frac{\frac{t^{n+1}}{t}}{(n+1)!}\,dt = \int_0^1 \frac{1}{(n+1)!} t^n\,dt = \frac{1}{(n+1)!} \frac{1}{n+1} t^{n+1} \Big|_0^1$$

$$= \frac{1}{(n+1)!} \cdot \frac{1}{n+1}$$

11.2.9

Finn $\int_0^1 \frac{1-e^{-t}}{t}\,dt$ med en nøyaktighet 10^{-3}

$a = 0$

$T_n e^x = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$

$R_n e^x = \frac{e^{c(x)}}{(n+1)!} x^{n+1}$

$c(x) \in (0, x)$ for $x > 0$.

$c(x) \in (x, 0)$ for $x < 0$.

Setter $x = -t$ for $t > 0$

$T_n e^{-t} = 1 - t + \frac{1}{2}t^2 + \dots + \frac{1}{n!}(-t)^n$

$R_n e^{-t} = \frac{e^{c(t)}}{(n+1)!}(-t)^{n+1}$ $\quad c(t) \in (-t, 0) \quad t > 0$

$$\frac{1-e^{-t}}{t} = \frac{1 - \left(1 - t + \frac{1}{2}t^2 + \dots + (-1)^n \frac{1}{n!} t^n + \frac{e^{c(t)}}{(n+1)!}(-t)^{n+1}\right)}{t}$$

$$= \frac{t - \frac{1}{2}t^2 + \dots + (-1)^{n+1}\frac{1}{n!}t^n + (-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^{n+1}}{t}$$

$\frac{1}{(n+1)!}\frac{1}{n+1} < \frac{1}{1000} \iff 1000 < (n+1)!(n+1)$

Holder for $n \geq 5$.