20th March, 2017

MAT-INF 3360

Mandatory assignment 2 of 2

Submission deadline

Thursday 6th of April 2017, 14:30 at Devilry (https://devilry.ifi.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (7th floor of Niels Henrik Abels hus, e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Consider the equation

$$\begin{aligned} u_t &= \varepsilon u_{xx} + u \quad \text{for } x \in (0,1), \quad t > 0, \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x) \end{aligned} \tag{1}$$

with a constant $\varepsilon > 0$.

a) Recall that a differential equation is considered stable if "a small change in the initial condition leads to small changes in the solution". To prove stability of equations like (1) it is sufficient to show that the solution $u(x,t)$ goes to 0 as $t \to \infty$. Observe that if we set $\varepsilon = 0$ in equation (1), it turns into a simple ordinary differential equation in t, and its solution will in general increase exponentially in t. Show that if $f(x) = \sin(\pi x)$ and

$$\varepsilon > \frac{1}{\pi^2}, \tag{2}$$

then the solution $u(x,t)$ of equation (1) goes to 0 as $t \to \infty$ for all $x \in [0,1]$.

b) Show that if $f(x) = \sin(k\pi x)$ for some positive integer k, and ε satisfies (2), then the solution $u(x,t)$ of equation (1) goes to 0 as $t \to \infty$ for all $x \in [0,1]$. Is it reasonable to expect this to be true for a more general function f? Explain why/why not.

Assume inequality (2) is satisfied in the remaining exercises.

c) Let

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \varepsilon \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2} + v_j^m, \quad j = 1,2,...,n, \quad m \geq 0, \tag{3}$$

with $v_0^m = v_{n+1}^m = 0$, and $v_j^0 = f(x_j)$, $j = 1,...,n$, be an explicit finite difference scheme for equation (1). Use von Neumann stability analysis to show that the conditions

$$\frac{\varepsilon \Delta t}{(\Delta x)^2} \leq \frac{1}{2}, \tag{4}$$

$$\varepsilon \geq \frac{1}{\pi^2 (1 - \frac{(\pi \Delta x)^2}{24})^2}, \tag{5}$$

implies that this scheme is stable.

d) Let

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \varepsilon \frac{v_{j-1}^{m+1} - 2v_j^{m+1} + v_{j+1}^{m+1}}{(\Delta x)^2} + v_j^{m+1}, \quad j = 1,2,...,n, \quad m \geq 0, \tag{6}$$

with $v_0^m = v_{n+1}^m = 0$, and $v_j^0 = f(x_j)$, $j = 1, ..., n$, be an implicit finite difference scheme for equation (1). Use von Neumann stability analysis to show that condition (5) implies that this scheme is stable.

e) Show that the implicit scheme (6) can be written in the form

$$Bv^{m+1} := ((1 - \Delta t)I + \varepsilon \Delta t A)v^{m+1} = v^m, \quad m \geq 0, \tag{7}$$

where $v^m = (v_1^m, ..., v_n^m)^T$ and the matrix A is given by equation (4.40) in the book. Explain why the above matrix B is positive definite for $\Delta t \leq 1$.

f) Implement the explicit scheme (3) for

$$f(x) = \begin{cases} 2x, & x \leq \frac{1}{2}, \\ 2(1 - x), & x > \frac{1}{2}. \end{cases} \tag{8}$$

Set $\varepsilon = 0.5$, choose a Δx and try experimenting with different time steps Δt to test condition (4). Plot your solutions at time $t = 0.1$.

g) Implement the implicit scheme (6), using formulation (7), with f given as in (8). Choose some Δx and Δt, and experiment with different ε to test condition (5). What happens when ε does not satisfy this condition? Plot your results at some "large enough" time t.