Hermite interpolation

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January 27, 2014

These notes extend the notion of Lagrange interpolation to Hermite interpolation. We study iterative interpolation and the Newton form.

1 Hermite interpolation

Suppose that $x_0, x_1, \ldots, x_n$ are distinct points in $[a, b]$ and that $f$ is a function that has derivatives of orders $0, 1, \ldots, r_i$, for each $i = 0, 1, \ldots, n$.

Theorem 1 With

$$N = n + \sum_{i=0}^{n} r_i,$$

there is a unique polynomial $p \in \pi_N$ such that

$$p^{(k)}(x_i) = f^{(k)}(x_i), \quad i = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, r_i. \quad (1)$$

Proof. Any $p \in \pi_N$ can be expressed uniquely as

$$p(x) = \sum_{j=0}^{N} c_j x^j,$$

and its $k$-th derivative is

$$p^{(k)}(x) = \sum_{j=k}^{N} \frac{j!}{(j-k)!} c_j x^{j-k}.$$

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The interpolation conditions (1) are then
\[ \sum_{j=k}^{N} \frac{j!}{(j-k)!} c_j x_i^{j-k} = f^{(k)}(x_i), \quad i = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, r_i. \]
which can be expressed as the linear system
\[ M c = f, \quad (2) \]
where \( c = (c_0, c_1, \ldots, c_N)^T \) and
\[
M = \begin{bmatrix}
1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^N \\
x_0 & 2x_0 & 3x_0^2 & 4x_0^3 & \cdots & N x_0^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_1 & 2x_1 & 3x_1^2 & 4x_1^3 & \cdots & N x_1^{N-1} \\
x_i & 2x_i & 3x_i^2 & 4x_i^3 & \cdots & N x_i^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots 
\end{bmatrix}, \quad f = \begin{bmatrix} f(x_0) \\ f'(x_0) \\ \vdots \\ f(x_1) \\ f'(x_1) \\ \vdots \end{bmatrix},
\]
and it is sufficient to show that \( M \) is non-singular. To demonstrate this suppose that \( c \) satisfies the homogeneous equation \( M c = 0 \). Then the polynomial
\[ q(x) = \sum_{j=0}^{N} c_j x^j \]
satisfies the conditions
\[ q^{(k)}(x_i) = 0, \quad i = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, r_i. \]
Then \( q \in \pi_N \) and has at least \( N + 1 \) roots, counting multiplicities, and, similar to the Lagrange case, by the fundamental theorem of algebra, \( q = 0 \). Hence \( c = 0 \) and \( M \) is indeed non-singular. \( \square \)

2 Iterative interpolation

One way of finding the Hermite interpolant \( p \in \pi_N \) is through the same iterative procedure we looked at in the first lecture. First, observe that if \( n = 0 \), the interpolant is the Taylor polynomial,
\[ p(x) = \sum_{j=0}^{r_0} f^{(j)}(x_0) \frac{(x - x_0)^j}{j!}. \quad (3) \]
Otherwise \( n \geq 1 \), and suppose that \( q \in \pi_{N-1} \) satisfies the conditions

\[
q^{(k)}(x_i) = f^{(k)}(x_i), \quad i = 0, 1, \ldots, n-1, \quad k = 0, 1, \ldots, r_i,
\]

\[
q^{(k)}(x_n) = f^{(k)}(x_n), \quad k = 0, 1, \ldots, r_n - 1,
\]

where we understand the second condition to be ‘empty’ if \( r_n = 0 \), and similarly suppose that \( r \in \pi_{N-1} \) satisfies

\[
r^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, 1, \ldots, r_0 - 1,
\]

\[
r^{(k)}(x_i) = f^{(k)}(x_i), \quad i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots, r_i.
\]

**Theorem 2** The polynomial

\[
p(x) := \frac{x_n-x}{x_n-x_0} q(x) + \frac{x-x_0}{x_n-x_0} r(x)
\]

is the polynomial in \( \pi_N \) that solves the Hermite interpolation problem (1).

**Proof.** By the Leibniz rule, the \( k \)-th derivative of \( p \) in (4) is

\[
p^{(k)}(x) = \frac{x_n-x}{x_n-x_0} q^{(k)}(x) + \frac{x-x_0}{x_n-x_0} r^{(k)}(x) + k \frac{r^{(k-1)}(x) - q^{(k-1)}(x)}{x_n-x_0},
\]

and it follows that

\[
p^{(k)}(x_i) = \frac{x_n-x_i}{x_n-x_0} q^{(k)}(x_i) + \frac{x_i-x_0}{x_n-x_0} r^{(k)}(x_i),
\]

for all \( i = 0, 1, 2, \ldots, n \) and \( k = 0, 1, \ldots, r_i \). Thus,

\[
p^{(k)}(x_0) = q^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, 1, \ldots, r_0,
\]

and,

\[
p^{(k)}(x_n) = r^{(k)}(x_n) = f^{(k)}(x_n), \quad k = 0, 1, \ldots, r_n,
\]

and, for \( i = 1, \ldots, n - 1 \),

\[
p^{(k)}(x_i) = \frac{x_n-x_i}{x_n-x_0} f^{(k)}(x_i) + \frac{x_i-x_0}{x_n-x_0} f^{(k)}(x_i) = f^{(k)}(x_i), \quad k = 0, 1, \ldots, r_i.
\]

\( \square \)
In order to look at an example, denote \( p \) by

\[
\begin{align*}
p_0, \ldots, 0, 1, \ldots, 1, \ldots, n, \ldots, n,
\end{align*}
\]

and consider cubic interpolation with \( x_0 = 0, x_1 = 1, \) and \( r_0 = r_1 = 1. \) The iteration gives

\[
\begin{align*}
p_{0011} &= (1 - x)p_{001} + xp_{011}, \\
p_{001} &= (1 - x)p_{00} + xp_{01}, \\
p_{011} &= (1 - x)p_{01} + xp_{11}, \\
p_{01} &= (1 - x)p_0 + xp_1,
\end{align*}
\]

and the Taylor polynomial (3) gives

\[
\begin{align*}
p_0 &= f(0), \quad p_{00} = f(0) + xf'(0), \\
p_1 &= f(1), \quad p_{11} = f(1) + (x - 1)f'(1).
\end{align*}
\]

Therefore,

\[
p(x) = (1 - x)^2(f(0) + xf'(0)) \\
+ 2x(1 - x)((1 - x)f(0) + xf(1)) \\
+ x^2(f(1) + (x - 1)f'(1)).
\]

### 3 The Newton form

A Hermite interpolant can also be represented in Newton form, the advantage being that the divided differences can be computed just once, and the evaluation of the interpolant for any given \( x \) is relatively fast, requiring only \( O(n) \) flops.

Defining the polynomial,

\[
\omega_N(x) := (x - x_0)^{r_0+1} \cdots (x - x_{i-1})^{r_{i-1}+1}(x - x_i)^{r_n},
\]

we can express the interpolant \( p = p_N \) as

\[
p_N(x) = p_{N-1}(x) + c_N\omega_N(x),
\]
with \( c_N \) the leading coefficient of \( p_N \). Continuing the recursion, we obtain the Newton form of the Hermite interpolant,

\[
p_N(x) = \sum_{i=0}^{N} c_i \omega_i(x).
\]  

(5)

In the special case that \( n = 0 \), the leading coefficient of \( p_N \) is

\[
c_N = f^{(r_0)}(x_0)/r_0!.
\]  

(6)

while if \( n \geq 1 \), the iterative interpolation algorithm, Theorem 2, gives a recursion for \( c_N \), because, with \( \text{lc}(p) \) denoting the leading coefficient of \( p \), equation (4) implies

\[
\text{lc}(p) = \frac{\text{lc}(r) - \text{lc}(q)}{x_n - x_0}.
\]  

(7)

In this way we can compute all the divided differences \( c_k \) required in (5), and due to (6) and (7) we now see that \( c_N \) is the divided difference

\[
c_N = \left[ x_0, \ldots, x_0, x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n \right] f.
\]

Consider again the example of cubic interpolation, with \( r_0 = r_1 = 1 \). The Newton form of the interpolant is

\[
p(x) = [x_0] f + [x_0, x_0] f (x - x_0) + [x_0, x_0, x_1] f (x - x_0)^2 + [x_0, x_0, x_1, x_1] f (x - x_0)^2(x - x_1).
\]

These divided differences can be computed from

\[
[x_0] f = f(x_0), \quad [x_1] f = f(x_1),
\]

\[
[x_0, x_0] f = f'(x_0), \quad [x_0, x_1] f = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad [x_1, x_1] f = f'(x_1),
\]

\[
[x_0, x_0, x_1] f = \frac{[x_0, x_1] f - f'(x_0)}{x_1 - x_0}, \quad [x_0, x_1, x_1] f = \frac{f'(x_1) - [x_0, x_1] f}{x_1 - x_0},
\]

and

\[
[x_0, x_0, x_1, x_1] f = \frac{[x_0, x_1, x_1] f - [x_0, x_0, x_1] f}{x_1 - x_0}.
\]
From now on we can simplify notation and consider a sequence of points \( x_0, \ldots, x_n \) in \([a, b]\), that are distinct or not. For each \( i \), we let \( \rho_i \) be the \textit{left-multiplicity} of \( x_i \),

\[
\rho_i = |\{0 \leq j < i : x_j = x_i\}|,
\]
i.e., the number of points in the sequence \( x_0, \ldots, x_{i-1} \) that are equal to \( x_i \). The Hermite interpolant to \( f \) is then the unique polynomial \( p_n \in \pi_n \) such that

\[
p_n^{(\rho_i)}(x_i) = f^{(\rho_i)}(x_i), \quad i = 0, \ldots, n.
\]

We have shown that

\[
p_n(x) = p_{n-1}(x) + c_n \omega_n(x),
\]
where

\[
\omega_n(x) = (x - x_0) \cdots (x - x_{n-1}),
\]
and \( c_n \) is the divided difference of \( f \),

\[
c_n = [x_0, x_1, \ldots, x_n] f.
\]

This divided difference is symmetric in the points \( x_0, \ldots, x_n \), and so we may assume that \( x_i \leq x_{i+1} \), in which case

\[
[x_0, x_1, \ldots, x_n] f = \frac{[x_1, \ldots, x_n] f - [x_0, \ldots, x_{n-1}] f}{x_n - x_0}, \quad \text{if } x_0 < x_n,
\]
and

\[
[x_0, x_1, \ldots, x_n] f = f^{(n)}(x_0)/n!, \quad \text{if } x_0 = \cdots = x_n.
\]
The Newton form of \( p \) is

\[
p(x) = \sum_{i=0}^{n} [x_0, \ldots, x_i] f \omega_i(x),
\]
and its error is

\[
f(x) - p(x) = [x_0, \ldots, x_n, x] f (x - x_0) \cdots (x - x_n),
\]
and there is some \( \xi \) in the smallest interval containing \( x_0, \ldots, x_n \) and \( x \) such that

\[
[x_0, \ldots, x_n, x] f = \frac{f^{(n+1)}(\xi)}{(n+1)!}.
\]
As an example, we find the error of the cubic Hermite interpolant \( p_3 \) we studied previously, at the points \( x_0, x_1 \), with \( x_0 < x_1 \). For \( x \in [x_0, x_1] \), if \( f \in C^4[x_0, x_1] \), there is some \( \xi \in [x_0, x_1] \) such that

\[
e(x) := f(x) - p_3(x) = (x - x_0)^2(x - x_1)^2 \frac{f^{(4)}(\xi)}{4!}.
\]

Since

\[
\max_{x_0 \leq x \leq x_1} (x - x_0)^2(x - x_1)^2 = \frac{h^4}{16},
\]

where \( h = x_1 - x_0 \), we deduce that

\[
\max_{x_0 \leq x \leq x_1} |e(x)| \leq \frac{h^4 M}{384},
\]

where

\[
M = \max_{x_0 \leq y \leq x_1} |f^{(4)}(y)|.
\]

As a final remark, we note that by the common Newton form of both Hermite and Lagrange interpolation, we see that a Hermite interpolant is the limit of Lagrange interpolants as some points coalesce, provided \( f \) has sufficiently many derivatives.