Mandatory assignment — MAT-INF4300 — Fall 2015

Information: The assignment must be submitted before 14:30 on Thursday, October 22, 2015, at the reception of the Department of Mathematics, the 7th floor of Niels Henrik Abels hus, Blindern. To have a passing grade you must have satisfactory answers to at least 50% of the questions and have attempted to solve all of them.

Problem 1

a)

Suppose $u \in C^2(\mathbb{R}^n \times (0,\infty))$ solves the heat equation $u_t - \Delta u = 0$. Then show that the parabolic rescaled function

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t), \qquad \lambda > 0,$$

also solves the heat equation.

b)

Use the result from a) to show that also the function $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation.

c)

Let η be a convex and twice continuously differentiable function, and assume u solves the heat equation. Show that the function $v = \eta(u)$ is a subsolution, i.e., $v_t - \Delta v \leq 0$.

Problem 2

Consider the initial value (Cauchy) problem

$$\begin{aligned} &u_t - \Delta u + cu = f \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where c is a constant, $f \in C^{2,1}(\mathbb{R}^n \times [0,\infty))$, and $u_0 \in C(\mathbb{R}^n)$, with f, u_0 compactly supported.

a)

Write down an explicit formula for the solution $u(x,t)$ of (1). Verify that this solution candidate belongs to $C^{2,1}(\mathbb{R}^n \times (0,\infty))$, satisfies the PDE for $x \in \mathbb{R}^n$ and $t > 0$, and satisfies the initial condition in the sense that $u(x,t) \to u_0(x_0)$ as $(x,t) \to (x_0, 0)$.

b)

Suppose $f \equiv 0$ and $u \to 0$ as $|x| \to \infty$. Then use the energy method to show that for $t > 0$,

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq e^{-ct} \|u_0\|_{L^2(\mathbb{R}^n)}.$$

Use this result to show that there exists at most one solution of (1) with $u \to 0$ as $|x| \to \infty$.

Problem 3

Consider the nonlinear initial-boundary value problem

$$\begin{aligned}
&u_t - \Delta u = -u^3 \quad \text{in } \Omega \times (0,\infty),\\
&u = u_0 \quad \text{on } \Omega \times \{t = 0\},\\
&u = 0 \quad \text{on } \partial\Omega \times (0,\infty),
\end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and u_0 is a given continuous function. Assume that there exist twice continuously differentiable solutions to this problem.

Use the energy method to prove that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)}, \qquad t > 0.$$

Problem 4

Let Ω be a bounded open subset of $\mathbb{R}^n$. Show that the Hölder space $C^{0,\gamma}(\Omega)$, with exponent $\gamma \in (0,1]$, is a Banach space.

Problem 5

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Fix any bounded open set V that is strictly larger than Ω. Then show that there exists a bounded linear operator

$$E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$$

such that for each $u \in W^{1,\infty}(\Omega)$ the following properties hold:

1. $Eu = u$ almost everywhere in Ω;

2. $\operatorname{supp}(Eu) \subset V$;

3. $\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,\infty}(\Omega)}$,

where the constant C does not depend on u.

Problem 6

Consider the function

$$u(x) = [x - x_0|^{-\alpha}, \qquad x \in B(x_0, 1) := \{x \in \mathbb{R}^3 : |x - x_0| < 1\}, \quad x_0 \in \mathbb{R}^3,$$

where $\alpha > 0$ is a constant. Determine the weak derivative Du of u. For which values of α do we have $u \in W^{1,2}(B(x_0, 1))$.