MAT1100 - Grublegruppe
Extra Problems 10

Jørgen O. Lye

Finite groups

A group is said to be finite if the underlying set has finitely many elements. The number of elements is referred to as the *order* of the group.

Quick exercise

Find the order of $\mathbf{Z}/(p\mathbf{Z})$.

Discrete symmetry groups

Consider a regular n-gon. Let D_n be the group of symmetries of the n-gon. The cases $n = 1$ and $n = 3$ should illuminate the general case.

D_1

A 1-gon is simply an interval $[a, b]$. What you can do with this is to reflect along the middle, interchanging the two vertices. Let this operation be called μ. Clearly $\mu \circ \mu = e =$id. This is the only symmetry of a 2-gon, so $D_1 \cong \{\mu, e\}$. Convince yourself that this is isomorphic to $\mathbf{Z}/(2\mathbf{Z})$.

D_3

The regular 3-gon is a triangle with angles $\frac{\pi}{3}$. Name the vertices $1, 2$ and 3. Let the *ordered* set of vertices be $[1, 2, 3]$.

Rotations

Geometrically, what we can do is to rotate the triangle by an angle of $\frac{\pi}{3}$ around the centre of mass (the centre of the triangle). To establish a convention, we will assume the rotation is counter clockwise. Call this rotation ρ. Convince yourself that ρ, ρ^2 and $\rho^3 = e$ are the possible rotations.

Exercise

Show that $\rho([1,2,3]) = [3,1,2]$ and $\rho^2([1,2,3]) = [2,3,1]$.

Reflections

In addition to rotations there are reflections. Let μ_i be the reflection through the line through vertex i and the middle of the opposing line-segment in the 3-gon.

Exercise

Show that $\mu_1([1,2,3]) = [1,3,2]$, $\mu_2([1,2,3]) = [3,2,1]$ and $\mu_3([1,2,3]) = [2,1,3]$.

The symmetries the the 3-gon is given by permuting its vertices. Convince yourself that this is so.

Exercise

Argue that the set of rotations of D_3 constitute a subgroup isomorphic to $C_3 \cong \mathbf{Z}/(3\mathbf{Z})$. Recall that C_p are p'th roots of unity. Do the set of reflections make up a subgroup? Hint: compute $\mu_1\mu_2$ or even just μ_i^2 for some i.

Exercise

Is the group D_3 abelian? That is, do all the elements of D_3 commute. What is the order of D_3? Is D_3 isomorphic to $\mathbf{Z}/(p\mathbf{Z})$ for some p?

D_4

Study the symmetries of a regular 4-gon, i.e. a square. How many rotations? How many reflections? What is the order of the group? Is it abelian?

D_n

This is the general case. Show that (or at least convince yourself that)

- There are n rotations (including identity), n reflections,
- The rotations make up a subgroup $C_n \cong \mathbf{Z}/(n\mathbf{Z})$,
- The reflections do not constitute a subgroup,
- D_n is non-abelian for $n > 2$.
- D_n is *not* isomorphic to $\mathbf{Z}/(p\mathbf{Z})$ for any p when $n > 2$ (hint: see the point above).

Is D_{n-1} a subgroup of D_n?

Groups of permutations

Pick some integer $n \geq 1$ and look at *ordered* set of elements $[1, 2, \cdots n]$. The group S_n, called the symmetric group, is the group of all permutations of these n. It's perhaps clear that this group is made up of combinations of permutations of pairs.

S_2 and S_3

Let's look at a couple of concrete example. S_3 is the set of permutations of $[1, 2, 3]$. If you didn't already do so, argue that $S_3 \cong D_3$. Similarly, argue that $S_2 \cong D_2$.

S_4

Argue that this group has 24 elements and show that it is non-abelian. Is it true that $S_4 \cong D_4$?

S_n

Show that:

- The order is $n!$,
- The group is non-abelian,

- S_n is not isomorphic to D_n for $n > 3$ (an easy solution is to look at the order),
- $S_1 \subset S_2 \subset S_3 \subset \cdots S_n$ as subgroups.

A_n

Finally, there is an important subgroup of S_n that should be mentioned. Define the sign of a permutation to be $(-1)^p$ where p is the number of times a *neighbouring pair* of elements is interchanged. For instance the identity has sign 1, the permutation $[1, 2, 3] \to [2, 1, 3]$ has sign -1 and the permutation $[1, 2, 3] \to [2, 3, 1]$ has sign 1. Not that the sign of reflections is -1 and the sign of rotations is $+1$. This sign is sometimes also called the parity.

Permutations of sign $+1$ are called even permutations, those with sign -1 are called odd permutations.

Let $A_n \subset S_n$ be the even permutations. Show that this is a subgroup. Is the set of odd permutations a group?

Remark

There is a useful theorem which I will not prove which is called Lagrange's theorem. It simply states the following. Assume G is a finite group and H is a subgroup. Then the order of H divides the order of G.

An example of its use is to argue that since D_n has order $2n$ and D_m has order $2m$, then D_m is cannot be a subgroup of D_n unless $m \leq n$ and m divides n. In particular, D_{n-1} is not a subgroup of D_n for $n > 3$.