MAT1100 - Grublegruppe
Extra Problems 11

Jørgen O. Lye

Vector spaces

A vector space over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) is an abelian group V with group operation written as addition such that the following holds for all $\alpha, \beta \in \mathbb{F}$, $v, w \in V$

- $\alpha v \in V$
- $\alpha(v + w) = \alpha v + \alpha w$
- $(\alpha + \beta)v = \alpha v + \beta v$
- $0v = 0$

The last point deserves a remark. The left 0 is in $\mathbb{F}$ and the right 0 is the identity element of V. What you should of course think of is $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$ where you are allowed to multiply a vector by a scalar. These aren't the only examples however, as you shall soon see.

Inner products

An inner product on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ which satisfies the following properties:

- $\langle \alpha v, u \rangle = \overline{\alpha} \langle v, u \rangle$
- $\langle v, \alpha u \rangle = \alpha \langle v, u \rangle$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, v \rangle$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$

- $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$

The bars are complex conjugation. This is of course only relevant for complex numbers. The inner product should be thought of as the dot product.

It is down to convention if you want to impose $\langle \alpha u, v \rangle = \overline{\alpha} \langle u, v \rangle$ or $\langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle$. You need to complex conjugate one, so that

$$\langle \alpha u, \alpha u \rangle = \alpha \overline{\alpha} \langle u, u \rangle = |\alpha|^2 \langle u, u \rangle$$

Exercises

Vector space $\mathbb{R}^n$

Convince yourself that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, and that

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

is an inner product.

Vector space $\mathbb{C}^n$

Similarly convince yourself that $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ with inner product

$$\langle z, w \rangle = \sum_{k=1}^{n} \overline{z_k} w_k$$

$\mathbb{C}^n$ as a real vector space

Finally, argue that $\mathbb{C}^n$ is a vector space over $\mathbb{R}$ with the samme inner product as above.

Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$?

Function spaces

Now that we have some intuition for vector spaces, let's move on to less familiar ones. Let $C([a, b])$ be the set of continuous functions on $[a, b]$ with values in $\mathbb{F}$. I.e. $f \in C([a, b])$ means $f : [a, b] \to \mathbb{F}$. If you're worried what continuity means for complex functions, recall that the definition is the same as for real functions where $|f(x) - f(y)|$ uses the complex version of the absolute value.

Exercise

Show that $V = C([a,b])$ is a vector space over $\mathbb{F}$.

Inner product on function spaces

I claim the the following is an inner product on $C([a,b])$

$$\langle f, g \rangle = \int_a^b \overline{f}(x) g(x)\, dx$$

You of course get the dubious pleasure of verifying this.

Norms from inner products

Note that by definition of inner products, we can set

$$\|v\|^2 = \langle v, v \rangle$$

and get a norm. This is precisely what happens in the Euclidean case:

$$|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$$

Note also that

$$\langle f, f \rangle = \int_a^b |f|^2\, dx = \|f\|_2^2$$

if you recall the so-called L^p norms

$$\|f\|_p = \left(\int_a^b |f|^p\, dx \right)^{1/p}$$

Optional exercise

Check that you do indeed get a norm from an inner product.

Inner products from norms?

A natural next question to ask is of course whether all norms come from an inner product. The answer turns out to be no, and here is a way of seeing this. Let us work with vector spaces over $\mathbb{R}$, just to avoid having to complex conjugate for a bit. If $\|v\|^2 = \langle v, v \rangle$, then

$$\|v + u\|^2 = \langle v + u, v + u \rangle = \langle v, v \rangle + 2 \langle u, v \rangle + \langle u, u \rangle$$

and

$$\|v-u\|^2 = \langle v, v\rangle - 2\langle v, u\rangle + \langle u, u\rangle$$

Adding these gives

$$\|v+u\|^2 + \|v-u\|^2 = 2\|v\|^2 + 2\|u\|^2$$

The norm $\|\cdot\|_\infty$ will not satisfy this. Look at the space $C([-\pi, \pi])$ with norm $\|f\|_\infty = \sup_{x\in[-\pi,\pi]}\{|f(x)|\}$. Let $v = \cos^2(x)$ and $u = \sin^2(x)$ and show that the identity

$$\|v+u\|^2 + \|v-u\|^2 = 2\|v\|^2 + 2\|u\|^2$$

fails to hold.

As a double check, use the norm

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2\, dx$$

to check that the identity above holds in this norm for the same two functions, as it should since it comes from the inner product defined on functions.

Orthogonality

With an inner product, we say that two vectors u, v are orthogonal if $\langle u, v\rangle = 0$. In particular, we get

$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 = \|u\|^2 + \|v\|^2$$

This is Pythagoras' theorem.

Let us stay in the vector space $C([-\pi, \pi])$ a bit longer and see how orthogonality works there, using the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} \overline{f} g\, dx$$

Exercise

Start out easily and check that $\sin(x)$ and $\cos(x)$ are orthogonal. I.e. compute

$$\langle \sin(x), \cos(x)\rangle = \int_{-\pi}^{\pi} \sin(x)\cos(x)\, dx$$

and check that you get 0.

Exercise

Check that the functions e^{inx} and e^{imx} are orthognal unless $m = n$. Now it's important that you remember to complex conjugate!

Exercise

Recall the identities

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Use these to check that the functions $\sin(nx)$, $\cos(mx)$ are orthogonal for all m and n. Then check that $\sin(nx)$ and $\sin(mx)$ are orthogonal unless $m = n$. Finally, check that $\cos(nx)$ and $\cos(mx)$ are orthogonal unless $m = n$.