MAT1100 - Grublegruppe
Extra Problems 12

Jørgen O. Lye

This document will deal with an important class of groups, namely matrix groups. The reader is thus assumed to know the definition of a group. Quotient groups will not be needed here.

General Linear Group

The mother of all the matrix groups is called the general linear group, written $GL(n, \mathbb{F})$ or $GL_n(\mathbb{F})$. $\mathbb{F}$ is as before $\mathbb{R}$ or $\mathbb{C}$. This is simply the set

$$GL_n(\mathbb{F}) = \{\ n \times n \text{ invertible matrices with entries in } \mathbb{F}\}$$

Exercises

Show that with matrix multiplication this is indeed a group.

Which of the following matrices are in $GL_2(\mathbb{R})$? $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$.

Are any of them in $GL_2(\mathbb{C})$?

Special Linear Group

An important subgroup of the general linear group is the special linear group. It is defined as the matrices

$$SL_n(\mathbb{F}) = \{A \in GL_n(\mathbb{F}) \big| \det(A) = 1\}$$

To see that this is indeed a subgroup, you need to know first of all that A is invertible if and only if $\det(A) \neq 0$ (you will show this next semester), so that all matrices with determinant 1 will be invertible. Then you need to argue that if A and B are in $SL_n(\mathbb{F})$, then AB is as well.

The geometric interpretation is to follow.

Orthogonal Group

Recall that $\mathbb{R}^n$ has the standard Euclidean inner product

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = x_1 y_1 + \cdots x_n y_n$$

The orthogonal group $O(n)$ is the subset of $GL_n(\mathbb{R})$ which preserves this inner product, meaning $A \in O(n)$ if and only if

$$(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. This equation can be written

$$(A\mathbf{x})^T (A\mathbf{y}) = \mathbf{x}^T A^T A \mathbf{y} = \mathbf{x}^T \mathbf{y}$$

Here I have used that $(AB)^T = (B^T A^T)$.

Exercise (hard?)

Use that the above equations is to hold for all $\mathbf{x}$ and $\mathbf{y}$ to argue that this is only possible if

$$A^T A = 1$$

I.e. the identity matrix. Such matrices are called orthogonal matrices. We will see below what their interpretation is.

Special Orthogonal Group

The equation $A^T A = 1$ means $\det(A^T A) = 1 = \det(A^T)\det(A) = \det(A)^2$. So $\det(A) = \pm 1$. The special orthogonal group $SO(n)$ is the subgroup of $O(n)$ where all matrices have determinant 1:

$$SO(n) = \{A \in O(n) \big| \det(A) = 1\}$$

Quick exercise

Convince yourself that this is a group.

Do the orthogonal matrices with $\det(A) = -1$ form a group?

Exercise

Write $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ for some vectors $\mathbf{v}_i$. Show that the equation

$$A^T A = 1$$

is equivalent to

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

i.e. that a matrix is orthogonal if and only if its columns are orthonormal (meaning orthogonal and having length 1).

Unitary Group

Before I proceed to the geometric meaning of the above, I would like to mention the complex analogues of the above. The standard inner product on $\mathbb{C}^n$ can be written

$$\mathbf{z} \cdot \mathbf{w} = \overline{z}_1 w_1 + \cdots + \overline{z}_n w_n = \mathbf{z}^\dagger \mathbf{w}$$

where $A^\dagger = \overline{(A^T)}$, complex conjugation and transposition. This is called the Hermitian conjugate. Other names are (according to Wikipedia) conjugate transpose, Hermitian transpose, bedaggered matrix, and adjoint matrix.

Anyway, the point is that the unitary group $U(n)$ is defined to be all complex $n \times n$ matrices preserving the complex inner product:

$$(A\mathbf{z})^\dagger (A\mathbf{w}) = \mathbf{z}^\dagger A^\dagger A \mathbf{w} = \mathbf{z}^\dagger \mathbf{w}$$

You can argue exactly as in the real case to see that this implies

$$A^\dagger A = 1$$

Such matrices are called unitary.

Exercise

Argue that $U(n)$ is a group.

Argue then that $U(1) \cong \mathbb{S}^1 \subset \mathbb{C}$, where you notice that $z^\dagger = \overline{z}$ when $z \in \mathbb{C}$.

Special Unitary Group

From the equation $A^\dagger A = 1$ we get that $|\det(A)|^2 = 1$ (show this!). This is a complex number, so the solution is actually $\det(A) = e^{i\theta}$ for some angle. The subgroup of $U(n)$ where matrices have $\det(A) = 1$ is called the special unitary group $SU(n)$:

$$SU(n) = \{A \in U(n) \big| \det(A) = 1\}$$

Argue that this is a group!

Computations

Which of the following matrices are in $U(2)$? Are any in $SU(2)$?
$\begin{pmatrix} 1 & -1 \\ i & 1 \end{pmatrix}$, $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Geometric Interpretation

Special linear group

Note that the volume spanned by 3 vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ is $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Let $B = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ be the matrix with the 3 vectors as columns. Notice that by definition of matrix multiplication, $AB = (A\mathbf{a}, A\mathbf{b}, A\mathbf{c})$, and so

$$\det(A\mathbf{a}, A\mathbf{b}, A\mathbf{c}) = \det(AB) = \det(A)\det(B) = \det(A)\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

This equation says that if you multiply all vectors by the matrix A, then you scale the volume by $\det(A)$. Hence matrices in $SL_3(\mathbb{R})$ preserve volumes! This is actually true in general, as $\det(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is the (hyper) volume spanned by $\mathbf{x}_1, \cdots \mathbf{x}_n$ in $\mathbb{R}^n$.

Orthogonal group

Since $\det(A) = \pm 1$ for things in $O(n)$, these matrices preserve volumes as well (negative determinant means that the orientation is reversed). More can be said about these matrices however, namely that they are precisely the reflections and rotations in $\mathbb{R}^n$. Let us compute this for $n = 1$ and 2.

Exericise

Argue that $O(1) = \{\pm 1\}$ where the group operation is multiplication. Argue if you want that this is isomorphic to $\mathbb{Z}/(2\mathbb{Z})$.

Exercise (somewhat long)

Look at an arbitrary 2×2 matrix with real entries.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The condition

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives you 4 equations for the 4 numbers a, b, c, d. Use this to argue that we can write

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

or

$$A = \begin{pmatrix} -\cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

for some suitable angle θ. Hint: $x^2 + y^2 = 1 \implies x = \cos(\theta)$ and $y = \sin(\theta)$ for some angle θ. You will need to operate with two different angles as some point!

Exercise

Using the above, show that $A \in SO(2)$ means we can write

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Exercise

Look at what happens to the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to see that the matrix above does indeed rotate. In general it is true that $SO(n)$ is the group of all rotations in $\mathbb{R}^n$.

Exercise

Argue that

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are in $O(2)$. Are they reflections or rotations? What about

$$A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$SO(3)$ briefly

In 3 dimensions there are 3 axis about which you can rotate. Argue that a rotation about the x-axis can be written

$$A_x = \begin{pmatrix} 1 & 0 & \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Check that this has determinant 1. Rotations about the y-axis can be written

$$A_y = \begin{pmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$$

What does the matrix for rotations about the z-axis look like? Write down this matrix using an angle ψ.

If you want an arbitrary rotation in 3 dimensions, this can be achieved by rotating separately around the 3 axis. This corresponds to multiplying the 3 matrices above to get one big and messy 3 matrix. You can do this if you're interested, but the answer isn't very enlightening.