MAT1100 - Grublegruppe
Extra Problems 13

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This document will serve as a brief introduction to some aspects of non-Euclidean geometry in dimension 2. We will start with some intuitively defined notions of distance.

Warning: there are many new concepts and much heavy machinery introduced in rapid succession. Don't despair if it is somewhat overwhelming.

Riemannian Metrics

Consider two points (x_1, y_1), (x_2, y_2) in the plane. The distance between them is $\Delta s^2 = \Delta x^2 + \Delta y^2$. Given a curve $\gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$ the length of the curve is given by

$$\int ds = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

where $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$. The intuition is to say that as Δx and Δy become "infinitesimal" (a notion usually not rigorously defined nowadays), we can write

$$ds = \sqrt{dx^2 + dy^2}$$

If you allow yourself to think that you can divide and multiply by dt, this becomes

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

which upon integrating gives the above. The expression ds is called a metric. It is related to, but is not the same as a metric defined earlier when we talked about metric spaces.

Hyperbolic half-plane

Consider the set $\mathbb{H} = \{z = x + iy \in \mathbb{C} | y > 0\}$. It is called the Poincaré half-plane. This is where we will build a new geometry called hyperbolic geometry. As such, define a metric (in the sense of the above section) on $\mathbb{H}$ by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Note that this is fine as $y \neq 0$ on our set. The idea here to to have an inhomogeneous idea of lengths; the distance between points gets stretched for y close to 0.

Example

Consider the curve $\gamma(t) = (a, t + \epsilon)$. $a \in \mathbb{R}$ and $\epsilon > 0$. This describes the line (in the Euclidean sense) from (a, ϵ) to $(a, 1 + \epsilon)$. Let us compute its hyperbolic length:

$$\int ds = \int_0^1 \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y(t)} \, dt = \int_0^1 \frac{\sqrt{0^2 + 1^2}}{t + \epsilon} \, dt = \ln(t + \epsilon) \Big|_0^1 = \ln\left(\frac{1 + \epsilon}{\epsilon}\right)$$

Note that this diverges as $\epsilon \to 0$. This is in contrast to the Euclidean length which is of course 1.

Exercise

Compute the hyperbolic length of the curve

$$\gamma(t) = (\cos(\pi t), \sin(\pi t))$$

and show that it is infinite. You may either estimate the integral using the techniques of 9.5 or you can compute the integral. What does the curve represent? Note that $0 < t < 1$ to keep $\gamma(t)$ in $\mathbb{H}$.

Lines

Euclidean lines can be defined to be curves $\gamma(t)$ such that given two points $p, q \in \mathbb{R}^n$, then $\gamma(0) = p$, $\gamma(1) = q$, and there are no other curves having a shorter length than γ. "Lines are the shortest paths between points."

The next question is of course: what are hyperbolic lines? We will try to answer this question in the next section.

Calculus of Variations

The following machinery is due to Euler, Lagrange and others around that time. It was invented to solve a similar problem, namely: given 2 points on a wall, what curve connecting the 2 points will make a bead sliding without friction (influenced only by gravity) along the curve travel from one point to the other in the shortest amount of time?

Let us return to the problem at hand. Write the length of a hyperbolic curve as

$$\ell(\gamma) = \int ds = \int \frac{\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}}{y} dy$$

Here I have implicitly assumed that I will be allowed to write x as a function of y. Since this curve starts at (x_1, y_1) and ends at (x_2, y_2), the integral goes from y_1 to y_2 with the condition that $x(y_1) = x_1$, $x(y_2) = x_2$. What we're then looking for is a function $x = x(y)$ such that the integral

$$\int_{y_1}^{y_2} \frac{\sqrt{x'(y)^2 + 1}}{y} \, dy$$

becomes as small as possible. A way of handling such scenarios is as follows.

Assume $x(y)$ is extremal. Consider a *variation* $\epsilon(y)$ such that we get a new curve $x(y) + \epsilon(y)$. Due to the boundary conditions we need to have $\epsilon(y_1) = \epsilon(y_2) = 0$. Let us call the integrand $\frac{\sqrt{x'(y)^2+1}}{y} = L = L(x, x', y)$ (here L does not depend on x).

I will need a result from multivariable Taylor-expansions which says

$$L(x + \epsilon, \dot{x} + \dot{\epsilon}, y) = L(x, \dot{x}, y) + \epsilon \frac{\partial L}{\partial x}(x, \dot{x}, y) + \dot{\epsilon} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, y) + \cdots$$

Here $\cdots$ means higher derivative and higher orders of ϵ. $\dot{x}$ means (for now) $\frac{dx}{dy}$.

Let's integrate:

$$\int_{y_1}^{y_2} L(x + \epsilon, \dot{x} + \dot{\epsilon}, y) \, dy \approx \int_{y_1}^{y_2} L(x, \dot{x}, y) + \epsilon \frac{\partial L}{\partial x}(x, \dot{x}, y) + \dot{\epsilon} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, y) \, dy$$

The first term

$$\int_{y_1}^{y_2} L(x, \dot{x}, y) \, dy$$

is the length of the curve $\ell(\gamma)$. So the other 2 terms are changes in the length. Let's look at them.

$$\int_{y_1}^{y_2} \epsilon \frac{\partial L}{\partial x}(x, \dot{x}, y) + \dot{\epsilon} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, y) \, dy = \int_{y_1}^{y_2} \epsilon \left(\frac{\partial L}{\partial x}(x, \dot{x}, y) - \epsilon \frac{d}{dy} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, y) \right) dy$$

In the last step I have integrated by parts and used the conditions $\epsilon(y_1) = \epsilon(y_2) = 0$. See to it that you agree!

The idea is now to demand that this change vanishes since we assumed we were at an extremal point (point meaning curve $x(y)$). This is completely analoguous to the Calculus theorem saying that $f'(x) = 0$ at an extremal point. Hence we want

$$0 = \int_{y_1}^{y_2} \epsilon \left(\frac{\partial L}{\partial x}(x, \dot{x}, y) - \epsilon \frac{d}{dy} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, y) \right) dy$$

Since $\epsilon(y)$ is arbitrary (apart from the boundary conditions) we conclude (why?) that

$$\frac{\partial L}{\partial x}(x, \dot{x}, y) - \epsilon \frac{d}{dy} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, y)$$

This called the Euler-Lagrange equation. It is a necessary condition for a maximum or minimum. It is the central tool we're going to use. If the above derivation is confusing, simply accept the Euler-Lagrange equation.

Applying Euler-Lagrange

Recall that in our hyperbolic case,

$$L = \frac{\sqrt{\dot{x}^2 + 1}}{y}$$

We compute from this that

$$\frac{\partial L}{\partial x} = 0$$

Hence the Euler Lagrange equations are simply

$$\frac{d}{dy} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0$$

Hence

$$\frac{\partial L}{\partial \dot{x}} = c$$

for some constant c. Computing the partial derivative gives

$$\frac{\dot{x}}{y\sqrt{\dot{x}^2 + 1}} = c$$

Exercise

Show that if $c = 0$, $x(y) = k$ for some constant. What lines are these?

For $c \neq 0$ the differential equation has the solutions

$$x = \pm\sqrt{R^2 - y^2} + x_0$$

for some constants x_0 and R. Rewrite this to get

$$(x - x_0)^2 + y^2 = R^2$$

What is this geometrically?

Conclusion

The above is a sketch of the proof that the lines in hyperbolic geometry are vertical (Euclidean) lines parallel with the y-axis and parts of circle arcs where the circle has centre on the y-axis.

Exercise

With the above as lines, convince yourself that the Euclidean axiom about parallel lines fails in hyperbolic geometry with the above lines.

Euclidean lines again

As a test of our Lagrangian machinery, let us see that we do get the correct answer for Euclidean lines. I.e. let

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\dot{x}^2 + 1}dy$$

So the Lagrange function L is

$$L = \sqrt{\dot{x}^2 + 1}$$

Once again we compute

$$\frac{\partial L}{\partial x} = 0$$

So the Euler-Lagrange equation tells us that

$$\frac{d}{dy}\frac{\partial L}{\partial \dot{x}} = 0$$

or that

$$\frac{\partial L}{\partial \dot{x}} = c$$

for some constant c. Show that the solutions to this are straight lines $x(y) = ay + b$.

Euclidean Isometries

In geometry we're often interested in maps that preserve distances. Such maps are called isometries. In Euclidean geometry, these are translations, rotations and reflections. A way of seeing this is that if

$$(x', y') = (x + a, y + b)$$

is a translation (the prime does not denote a derivative here!) then $dx' = d(x + a) = dx$ and $dy' = dy$. So the Euclidean metric

$$ds^2 = dx^2 + dy^2$$

is invariant;

$$dx^2 + dy^2 = dx'^2 + dy'^2$$

For a rotation we had

$$(x', y') = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$$

hence (for a fixed angle θ!)

$$dx' = dx\cos(\theta) - dy\sin(\theta)$$

$$dy' = dx\sin(\theta) + dy\cos(\theta)$$

From which we get

$$(dx')^2 = dx^2\cos^2(\theta) - 2dxdy\sin(\theta)\cos(\theta) + dy^2\sin(\theta)$$

and

$$(dy')^2 = dx^2\sin^2(\theta) + 2dxdy\sin(\theta)\cos(\theta) + dy^2\cos^2(\theta)$$

Combining gives

$$(dx')^2 + (dy')^2 = dx^2 + dy^2$$

So rotations preserve the Euclidean distance ds^2.

Quick calculation

Show that the reflections $(x', y') = (-x, y)$ and $(x', y') = (x, -y)$ are isometries.

Hyperbolic isometries

We have the expression

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

This can be written as

$$ds^2 = \frac{dz d\overline{z}}{y^2}$$

with $z = x + iy$, $dz = dx + idy$ and $d\overline{z} = dx - idy$. I claim that the map

$$z' = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ will map $\mathbb{H} \to \mathbb{H}$ and be an isometry. Justify the following claims:

$$dz' = \frac{1}{(cz + d)^2} dz$$

$$dz' d\overline{z}' = \frac{dz d\overline{z}}{|cz + d|^4}$$

$$y' = \text{Im}\left(\frac{az + b}{cz + d}\right) = \frac{y}{|cz + d|^2}$$

$$\frac{dz' d\overline{z}'}{y'^2} = \frac{dz d\overline{z}}{y^2}$$

Hence these are indeed isometries.

Fractional linear transformations

Let's study functions of the form

$$f(z) = \frac{az + b}{cz + d}$$

$a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ some more. They are called fractional linear transformations. Note that if

$$g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

Then

$$f(g(z)) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\delta)z + (d\beta + d\delta)}$$

Hence the composition of two fractional linear transformations is again a fractional linear transformation. The function $f(z) = z$ is the identity, so with compositions of functions as the operation, the set of all fractional linear transformations is a group!

If we think of f as being given by the 4 numbers $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and g as being given by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then the function $f(g(z))$ is given by the numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

The requirement $ad - bc = 1$ says precisely that the determinant is 1. So in the language of group theory, there is a homomorphism from $SL_2(\mathbb{R})$ to the FLT's (fractional linear transformations). It is clearly surjective (why?). Is it injective? Well, note that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, but that $\frac{az+b}{cz+d} = \frac{(-a)z+(-b)}{(-c)z+(-d)}$. It can be shown (feel free to try) that this is the only thing standing in the way of injectivity. Hence the group of fractional linear transformations is isomorphic to $SL_2(\mathbb{R})/\mu_2$ where $\mu_2 = \{\pm 1\}$.

Remarks

Hyperbolic Geometry

This is just scratching the surface of hyperbolic geometry. You can define triangles using the above lines, then define angles. You can move triangles around and ask when two triangles are the same modulo a fractional linear transformation (i.e. an isometry). The conclusion would then have been that given the 3 angles of the triangle, the lengths of the sides would have been completely determined!

We could also have defined areas, and then shown that a hyperbolic trangle consisting of edges that are circle arcs starting and stopping at $y = 0$ would have an area of π, even though the edges have infinite length.

Finally, you can start asking what the fractional linear transformations do and start classifying them.

Calculus of Variations

The Calculus of Variations goes a lot deeper than just hyperbolic geometry. It is used a lot both in theoretical physics and in finance.

Physics

In physics, a basic Lagrangian is $L(x(t), \dot{x}(t), t) = T - V$ where $x(t)$ is the position of a particle at time t, T is the particle's kinetic energy, and $V = V(x)$ is the potential energy. If you use $T = \frac{1}{2}m\dot{x}^2$ as the kinetric energy, the Euler-Lagrenge equations become

$$V'(x) + m\ddot{x} = 0$$

or

$$m\ddot{x} = -V'(x)$$

A standard definition of a potential $V(x)$ is that $F = -V'(x)$ where F is the force. So the Euler-Lagrange equations reproduce Newton's second law of motion, $F = ma = m\ddot{x}$.

You can actually deduce the basic equations of quantum field theory, general relativity, and string theory from suitable Lagrange functions (albeit in more variables).

Finance

In finance, L is typically some cost function you want to minimize or it's a utility function you want to maximize. Typically however, you need to impose more constraints in finance than I have done here; if one of the variables in the Lagrange function is supposed to represent e.g. money or time, then both should be taken as finite for the answer to be have applications in the real world. Then you're headed into the domain of *control theory.* Both calculus of variations and control theory (with toy models from finance as examples) are part of the course MAT2440.