MAT1100 - Grublegruppe
Extra Problems 14

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This note will deal with some basic Fourier-theory. We will here just assume convergence, even though this is a somewhat subtle and deep question.

The basic statement

Assume f is a (piecewise) continuous function on $f : [-\pi, \pi] \to \mathbb{C}$. Piecewise continuous means that by removing finitely many points from $[-\pi, \pi]$, f is continuous. The idea behind Fourier-theory is to write

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \tag{1}$$

for some coefficients $c_n \in \mathbb{C}$. Alternatively, this could be written

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{2}$$

If f is real-valued, we can choose the coefficients a_n and b_n to be real numbers. Like I said above, we will assume we can write f as such a sum, and that the equality holds expect possibly at isolated points.

Quick question

Assume either equation 1 or 2 holds for a function f. Argue that we then need to have $f(x + 2n\pi) = f(x)$ for any n. I.e. that f is periodic when extended to all of $\mathbb{R}$ (f was originally only defined on $[-\pi, \pi]$).

Finding the coefficients

You were supposed to show earlier that

$$\int_{-\pi}^{\pi} e^{-imx} e^{inx}\, dx = 2\pi\delta_{mn} = \begin{cases} 2\pi & m = n \\ 0 & n \neq m \end{cases}$$

Use this to argue that

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = 2\pi c_m$$

I.e. that if we still assume the expansion of equation 1, we can find c_n by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

If you recall the L^2 inner-product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} \overline{f} g\,dx$$

you will see that the above can be written as follows. Let $e_n(x) = e^{inx}$. Then

$$c_n = \frac{1}{2\pi}\langle e_n, f\rangle$$

And as such,

$$f(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \langle e_n, f\rangle\, e_n$$

This is analoguous to writing a vector in $\mathbb{R}^n$ as

$$\mathbf{v} = \sum_{k=1}^{n} (\mathbf{v}\cdot\mathbf{e}_k)\mathbf{e}_k$$

Since there are n basis vectors $\mathbf{e}_k$ for $\mathbb{R}^n$, we say that $\mathbb{R}^n$ is an n-dimensional vectors space. If you accept that the set $\{e_n(x)\}_{n\in\mathbb{Z}}$ serves the same purpose for $L^2([-\pi,\pi])$, then you will hopefully agree that $L^2([-\pi,\pi])$ is infinite-dimensional as a vector space.

Example

Let us test this newfound technology. Let $f(x) = x$. Let's try computing its Fourier-coefficients to see how this work.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx$$

We solve this by integrating by parts ("delvis integrasjon"). For $n \neq 0$, we have

$$c_n = \frac{1}{2\pi}\left(-\frac{xe^{-inx}}{in}\Big|_{-\pi}^{\pi} - \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx\right) = \frac{1}{2\pi}\left(\frac{i}{n}(\pi e^{in\pi} - (-\pi)e^{-in\pi} - 0\right)$$

$$c_n = \frac{i}{2n\pi}\,(2\pi(-1)^n) = \frac{i}{n}(-1)^n$$

For $n = 0$, we get that

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

So if Fourier-theory is supposed to work, we should be able to write

$$f(x) = x = \sum_{n\neq 0} \frac{i}{n}(-1)^n e^{inx}$$

We can work a bit on the sum:

$$x = \sum_{n=1}^{\infty} \frac{i}{n}(-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{i}{-n}(-1)^{-n} e^{-inx} = i\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(e^{inx} - e^{-inx})$$

and finally

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$$

Exercise

Go through the above computation and justify the steps!

The plots of the sum

$$\sum_{n=1}^{N} \frac{2(-1)^{n+1}}{n}\sin(nx)$$

are included for $N = 2, 5, 15,$ and 50. It gives numerical "evidence" that Fourier theory might work. Can you think of what happens at the endpoints? Hint: Note that the Fourier-series is periodic with period 2π and $f(x) = x$ is not!

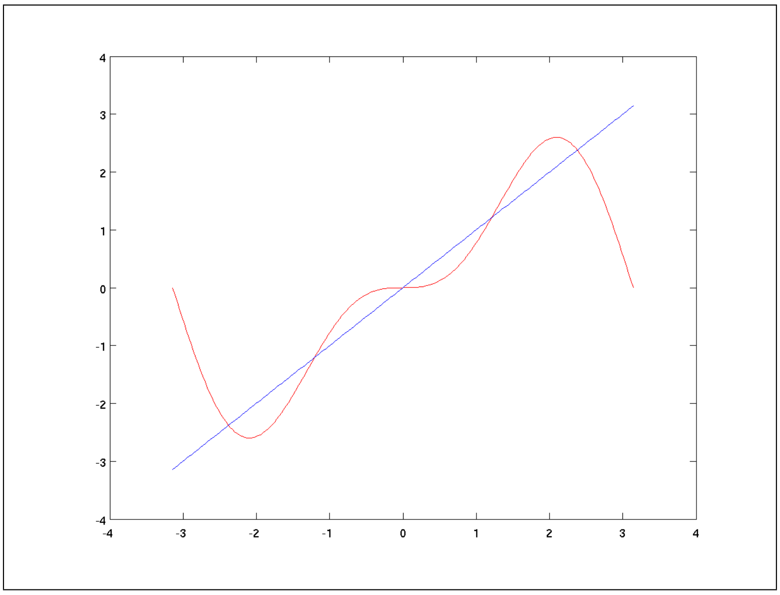

Figure 1: Plot showing $f(x) = x$ (blue line) and its truncated Fourier series with $N = 2$ terms (red).

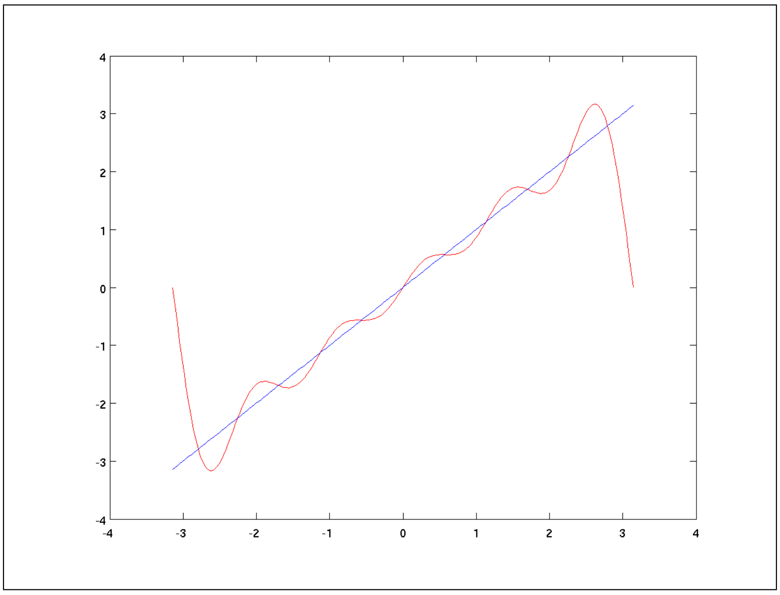

Figure 2: Plot showing $f(x) = x$ (blue line) and its truncated Fourier series with $N = 5$ terms (red).

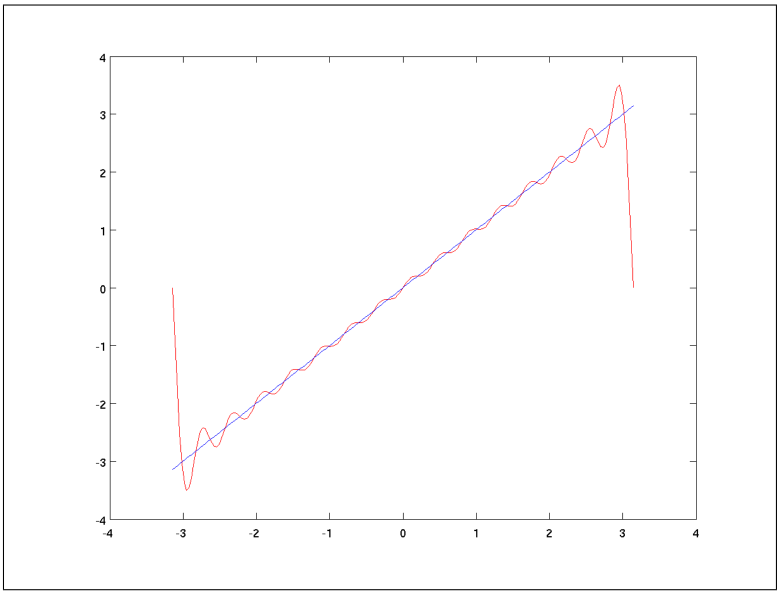

Figure 3: Plot showing $f(x) = x$ (blue line) and its truncated Fourier series with $N = 15$ terms (red).

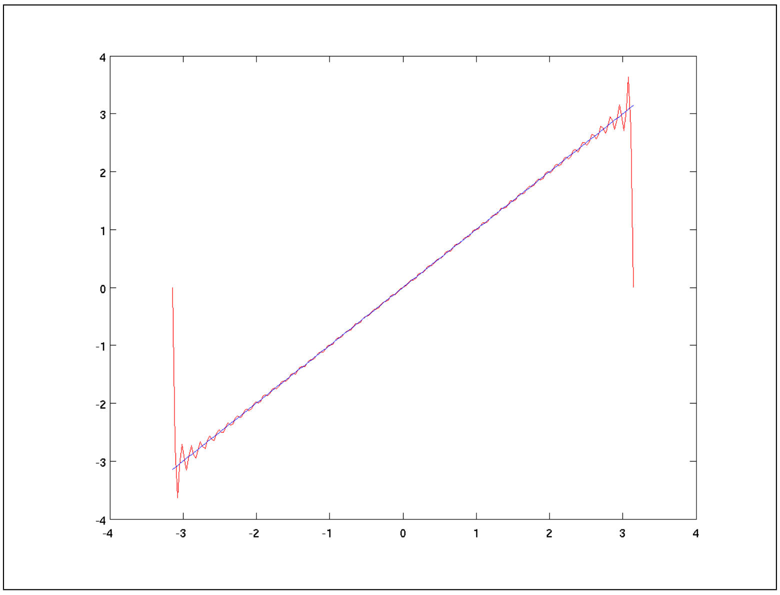

Figure 4: Plot showing $f(x) = x$ (blue line) and its truncated Fourier series with $N = 50$ terms (red).

Example

Let's do another example! Let's try $f(x) = x^2$. We need to compute

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} \, dx$$

This is done by integration by parts again, and is left to the reader. The answer is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Note that for this example, $c_0 \neq 0$.

Plots are included for $N = 2, 5$, and 10. Notice how quickly this converges. Note also that the endpoints don't blow up in this example. Any comment? In particular, think about how $f(x) = x^2$ looks when extended periodically.

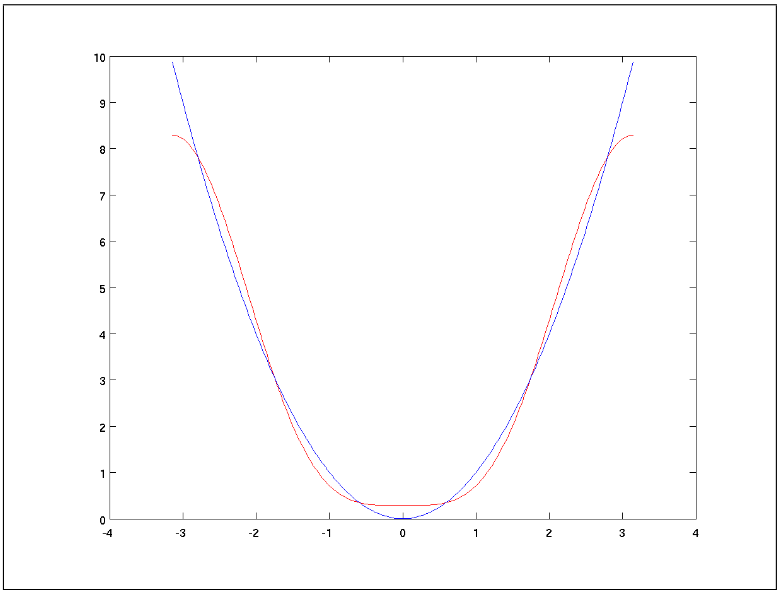

Figure 5: Plot showing $f(x) = x^2$ (blue curve) and its truncated Fourier series with $N = 2$ (red curve).

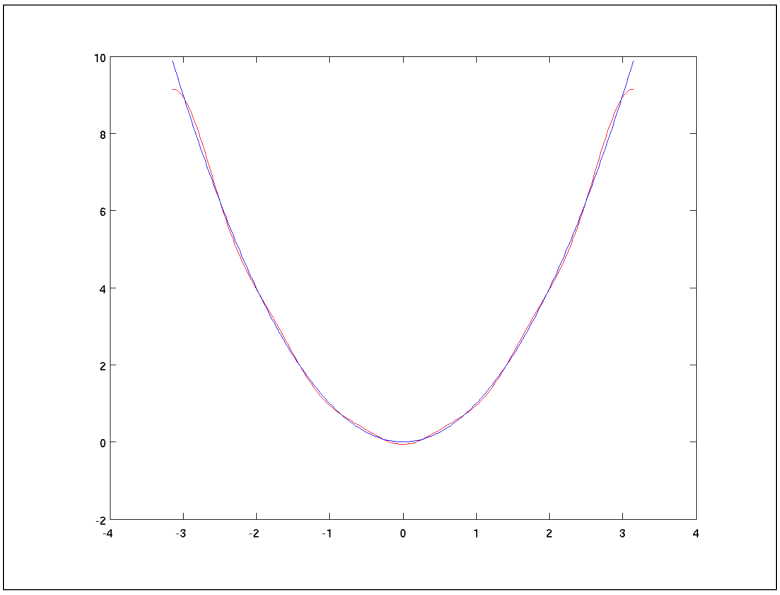

Figure 6: Plot showing $f(x) = x^2$ (blue curve) and its truncated Fourier series with $N = 5$ (red curve).

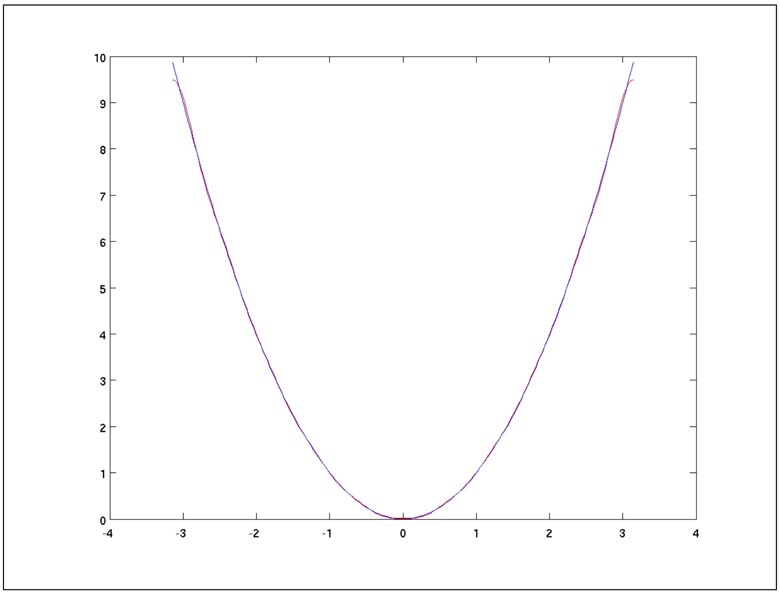

Figure 7: Plot showing $f(x) = x^2$ (blue curve) and its truncated Fourier series with $N = 10$ (red curve).

We can get some neat little result out of the Fourier-series of x^2. If we believe in its convergence at $x = \pi$, we get

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(n\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Rearranging this gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

If you recall that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

then this seems to show that $\zeta(2) = \frac{\pi^2}{6}$.

If you want $\zeta(4)$, you could try computing the Fourier series of x^4, but it quickly gets tedious.

Exercise

Notice that we could write

$$x = \sum_{n=1}^{\infty} a_n \sin(nx)$$

and

$$x^2 = \sum_{n=0}^{\infty} b_n \cos(nx)$$

Look at what happens when $x \mapsto -x$ and use this to argue that

$$x^{2n} = \sum_{n=0}^{\infty} b_n \cos(nx)$$

(no sines) and

$$x^{2n+1} = \sum_{n=1}^{\infty} a_n \sin(nx)$$

(no cosines). More generally: If $f(-x) = f(x)$ for all x, then

$$f(x) = \sum_{n=0}^{\infty} b_n \cos(nx)$$

If $f(-x) = -f(x)$ for all x, then

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

Finally, a somewhat unrelated question. Show that a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(-x) = -f(x)$ for all x must satisfy $f(0) = 0$. Convince yourself that this is consistent with the claim I make about its Fourier series above.