MAT1100 - Grublegruppe
Extra Problems 2

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In this note we look at an important application of sequences (følger), namely sequences of functions. For each $n \geq 0$, let $f_n(x)$ be a function. An example could be $f_n(x) = x^n$, so the sequence is $1, x, x^2, x^3, \cdots$. We could also consider sums of such functions called series (rekker), like a geometric series

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

where we for the time being assume the limit

$$\lim_{N \to \infty} \sum_{n=0}^{N} x^n$$

exists.

Exercise 1

Show that for $z \in \mathbb{C}$ with $|z| < 1$, the geometric series

$$\sum_{n=0}^{N} z^n$$

converges to

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}$$

Hint: Multiply by $(1 - z)$ and observe why the resulting sum is called telescoping.

Differentiation and integration

It can be shown (but we will not do so here!) that for series of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

can, with c_n being numbers, be differentiated and integrated term by term:

$$\int \left(\sum_{n=0}^{\infty} c_n x^n \right) dx = \sum_{n=0}^{\infty} \left(\int c_n x^n \, dx \right) = \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$$

And

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} c_n x^n \right) = \sum_{n=0}^{\infty} \frac{d}{dx} c_n x^n = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

Exercise 2

Differentiate the power series defining e^x to show that $(e^x)' = e^x$. I.e. differentiate

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(Note: in the above notation we have $c_n = \frac{1}{n!}$).

Exercise 3

Integrate the geometric series to show that

$$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

for $|z| < 1$. What happens to the left hand side when you try putting $z = 1$?

Note: this is actually the correct value, hinting that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Exercise 4

It can be shown (but again, we will not do so) that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

Converges for all $s > 1$. We saw in Exercise 3 that it seems to be infinite for $s = 1$, so we will keep to $s > 1$. This defines the Riemann-Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Amongst the fun results that can be shown using Fourier analysis is that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$$

What you will instead show (more or less) is that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \cdots$$

It might be easier to show the exact same thing written differently, namely that

$$\zeta(s) \prod_{p \text{ prime}} (1 - \frac{1}{p^s}) = 1$$

Follow the induction steps

a)

Show that (or argue that)

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = \left(1 - \frac{1}{2^s}\right) \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots\right)$$

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \cdots - \frac{1}{2^s} - \frac{1}{4^s} - \frac{1}{6^s} \cdots = \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{1}{n^s}$$

The symbol $n \nmid m$ means "n does not divide m", whereas $n \mid m$ means "n divides m".

b)

Similarly, argue that

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\zeta(s)=\sum_{\substack{n=1\\2,3\nmid n}}^{\infty}\frac{1}{n^s}$$

c)

Argue loosely or by induction that continuing this way gives the desired formula

$$\zeta(s)=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$$