MAT1100 - Grublegruppe
Extra Problems 3

Jørgen O. Lye

Metric Spaces

A natural generalisation of the concepts of convergence and continuity is to work with something called metric spaces. They are defined as follows

Definition

A metric space is a set X along with a function (called a metric) $d : X \times X \to \mathbb{R}$ which satisfies the following conditions for all $x, y, z \in X$:

- $d(x, y) = d(y, x)$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) \leq d(x, z) + d(z, y)$

These are known as being symmetric, positive definite, and obeying the triangle inequality.

Examples

Standard examples

The prototypical examples are $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$ and $X = \mathbb{C}$ with $d(z, w) = |z - w|$.

These work on $\mathbb{R}^n$ and $\mathbb{C}^n$ as well:

$$d((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots y_n)) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

for $\mathbb{R}^n$ and

$$d((z_1, \cdots, z_n), (w_1, \cdots w_n)) = \sqrt{\sum_{k=1}^{n} |z_k - w_k|^2}$$

for $\mathbb{C}^n$.

Manhattan metric

$X = \mathbb{R}^2$ with the metric defined as $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ Note that this is not the standard Euclidean metric!

Discrete metric

Let X be any set and define

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Function space examples

These examples a bit more complicated, as the set X consists of functions.

Supremum metric

Let X be the set of continuous functions on $[a, b]$, often written $X = C([a, b])$. For $f, g \in X$ (2 continuous functions), define

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

L^p-metric

I will not here explain the name L^p (read "L p"), but let $X = C([a, b])$ be as above and define

$$d_p(f, g) = \left(\int_a^b |f(x) - g(x)|^p \, dx \right)^{1/p}$$

$p = 1$ and $p = 2$ are the most frequently seen of these spaces.

Exercises

Checking axioms

Check that all the above metric spaces are indeed metric spaces. I.e. check that they satisfy the 3 points in the definition of a metric. The triangle inequality will be the hardest. For the L^p-spaces you should be content with showing the triangle inequality for $p = 1$. You will need some additional computational technology for $p > 1$.

Balls

Let X be a metric space. For $x_0 \in X$ some point, define

$$B(x_0, R) = \{x \in X \big| d(x, x_0) < R\}$$

Then $B(x_0, R)$ is called the open ball centred at x_0. Try to convince yourself that this is indeed a ball in $\mathbb{R}^3$. What does it look like in $\mathbb{R}^2$ and $\mathbb{R}$ with the standard metric?

What does the ball look like with the Manhattan metric?

What about the discrete metric?

Convergence

Let $\{x_n\}$ be a sequence in X. One says that $x_n \to x$ if given $\epsilon > 0$ there is an $N > 0$ such that $d(x, x_n) < \epsilon$ for all $n \geq N$.

Convergence in discrete metric

Show that a sequence converges in the discrete metric if and only if the sequence is eventually constant. This means that there is an n such that $x_n = x_{n+m}$ for all $m \geq 0$.

Uniqueness of limits

Let X be a metric space. Use the axioms to show that if $x_n \to a$ and $x_n \to b$, then $a = b$. I.e. the limit of a sequence (if it exists) is unique.

Continuity

Let X and Y be metric spaces with metrics d_X and d_Y respectively. Let $f : X \to Y$ be a function. Then f is continuous at $x_0 \in X$ if for any $\epsilon > 0$ you can find a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$.

Continuity and limits

Show that a function $f : X \to Y$ between metric spaces is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ (in Y!) for any sequence x_n converging to x_0 (in X!).