MAT1100 - Grublegruppe
Extra Problems 4

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Cauchy sequences

A sequence $\{x_n\}$ is called a Cauchy sequence (Cauchy-følge) if given an $\epsilon > 0$, there is an $N > 0$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.

If you read the note about metric spaces, replace $|x_n - x_m|$ by $d(x_n, x_m)$ and you have the definition of a Cauchy sequence in a metric space.

Problem 1

Show that if $\{x_n\}$ is a convergent sequence then $\{x_n\}$ is a Cauchy sequence.

Problem 2

Consider the set $\mathbb{Q}$ of rational numbers, and let $\{x_n\}$ be the decimal expansion of $\sqrt{2}$ where x_n has n decimals included. By this I mean

$$x_1 = 1.4 = \frac{14}{10}$$

$$x_2 = 1.41 = \frac{141}{100}$$

$$x_3 = 1.414 = \frac{1414}{1000}$$

et cetera. Then $x_n \in \mathbb{Q}$ for all $n \geq 1$. Show that it is a Cauchy sequence, where the distance between rational numbers is defined in exactly the same way as for real numbers.

Problem 3

Show that $\sqrt{2} \notin \mathbb{Q}$.

Hint: Assume it is, i.e. $\sqrt{2} = \frac{a}{b}$. Demand that a and b do not have common factors. Squaring shows that $2 = \frac{a^2}{b^2}$, showing that b^2 divides a^2. Show that this means b divides a, contradicting the demand that a and b do not have common factors.

Problem 4

Conclude that the sequence in Problem 2 is a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$.

Completeness

A metric space is called complete if every Cauchy sequence converges. The metric spaces $\mathbb{R}$ and $\mathbb{C}$ are complete with their usual metrics, as are $\mathbb{R}^n$ and $\mathbb{C}^n$ for any $n \geq 0$. The metric space $\mathbb{Q}$ with the same metric as $\mathbb{R}$ is not complete.

Recall that I defined $L^1(C[a, b])$ as the metric space of continuous functions defined on $[a, b]$ with the metric

$$d(f, g) = \int_a^b |f(x) - g(x)|\, dx$$

You will now show that the space $L^1(C[0, 1])$ is *not* complete.

a

Let

$$f_n(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ e^{-n\left(x - \frac{1}{2}\right)} & x \in [\frac{1}{2}, 1] \end{cases}$$

and compute

$$d(f_n, f_m) = \int_0^1 |f_n - f_m|\, dx$$

b)

Show (or argue) that given an $\epsilon > 0$, you can find an $N > 0$ such that $d(f_n, f_m) < \epsilon$ whenever $m, n \geq N$. In other words, show that f_n is a Cauchy sequence.

c)

Show that f_n converges to

$$f(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ 0 & x \in [\frac{1}{2}, 1] \end{cases}$$

in the metric of $L^1(C[0,1])$. I.e. show that

$$\int_0^1 |f_n(x) - f(x)|\, dx \to 0$$

as $n \to \infty$.

d)

Since f is obviously not a continuous function (you can show it if you like), $f \notin L^1(C([0,1]))$.

Conclude that $L^1(C([0,1]))$ is not complete.