MAT1100 - Grublegruppe
Extra Problems 5

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Topological notions

This note will deal with some topological notions. If you read the metric space notes, you can let X be a metric space with some metric d. If not, let the space be $\mathbb{R}$ and use $d(x, y) = |x - y|$ as the distance between points in all the formulations below.

Open sets

A subset $U \subset X$ is called open if for any $p \in U$, there is a ball $B_R(p) = \{x \in X \big| d(x, p) < R\}$ such that $B_R(p) \subset U$. The intuition is that all points in an open set have some neighbourbood also contained in the open set.

Exercise

Let $x \in X$ be any point and pick some $k > 0$. Let $A = B_k(x)$. Show that this is an open set. In words; show that an open ball is open. Note: this is not as obvious as it sounds!

Limit points and closed sets

Let $V \subset X$ be some set. A point $x \in X$ is called a *limit point* of A if any ball $B_R(x)$ intersects both A and $A^c = X \setminus A$ (the complement of A). Intuitively a limit point lies on the boundary of A.

A set V is called *closed* if it contains all its limit points.

Exercise

Show that a set V is closed if and only if $X \setminus V$ is an open set.

Hint: show that a set is open if and only if it contains none of its limit points.

Exercise

Show that the complement of a closed set is open.

Open versus closed

A set can be both open and closed. The entire space X and the empty set $\emptyset$ are both open and closed. One way of seeing that is for instance that X doesn't have any limit points, and as such it contains all its limit points (i.e. none). Hence it is closed. Clearly any ball $B_R(x)$ fits inside X per definition of the ball, so X is open. Taking complements shows that the empty set is open.

Exercise

Show that an arbitrary union of open sets is open and a finite intersection of open sets is open.

Continuity

In Calculus, we say that a function $f : U \to \mathbb{R}$ ($U \subset \mathbb{R}$) is continuous at a point x_0 if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

In a metric space, I previously defined continuity of a function $f : X \to Y$ to be that given $\epsilon > 0$, there is a $\delta > 0$ such that

$$d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \epsilon$$

In this note you will learn about an equivalent notion.

Inverse image

Recall first that the *inverse image* of a function f means

$$f^{-1}(U) = \{x \in X \Big| f(x) \in U\}$$

For instance if $X = \mathbb{R}$ and $U = (-1, 1)$ with $f(x) = e^x$ then $f^{-1}(U) = (-\infty, 0)$ while if $f(x) = \sin(x)$ then $f^{-1}(U) = \mathbb{R} \setminus \bigcup_{n=-\infty}^{\infty} \{\frac{\pi}{2} + n\pi\}$.

Exercise

Let $X = \mathbb{R}$ and $U = (1, 2)$. Find the inverse images of the functions $f(x) = x^2$, $f(x) = \cos(x)$, $f(x) = x + 1$.

Continuity via open sets

Let $f : X \to Y$ be some function between metric spaces. We say that the function is continuous at a point $x_0 \in X$ if for any open set $V \subset Y$ with $f(x_0) \in V$ there is an open set $U \subset X$ with $x_0 \in U$ such that $f^{-1}(V) \subset U$.

More simply put: for any open subset $V \subset Y$, $f^{-1}(V) \subset X$ is open.

Exercise

Show that a function is continuous with this definition if and only if it is continuous using the metric space (or Calculus) definition.

Hint: If $d(x_0, x) < \delta$ it means that $x \in B_\delta(x_0)$. If $d(f(x), f(x_0)) < \epsilon$ then $f(x) \in B_\epsilon(f(x_0))$.