MAT1100 - Grublegruppe
Extra Problems 6

Jørgen O. Lye

Topology

This note extends note 5 and continuous the development of topological concepts. We saw in note 5 that you can define continuity as soon as you know which sets are open. For a metric space, we used balls to define open sets, so we referenced the metric. We will now assume we know which sets are open and forget about the metric. This leads to the notion of a topological space.

Topological space

Let X be some set along with a collection of subsets $\mathcal{T}$, i.e. $U \in \mathcal{T} \implies U \subset X$ such that the following properties hold

- $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
- If some collection of open sets U_i, $i \in I$ are in $\mathcal{T}$ the $\bigcup U_i \in \mathcal{T}$
- If $U_1, \cdots U_n$ are in $\mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$.

Then X is called a *Topological space* The sets in $\mathcal{T}$ are called the open sets.

Exercise

Argue, using the results of Note 5, that any metric space is a topological space.

Exercise

Show that any set X with $\mathcal{T} = \{X, \emptyset\}$ is a topological space. We then say that X has the *trivial topology.*

Exercise

Show that any set X with $\mathcal{T} = \{\text{All subsets of } X\}$ is a topological space. This choice of $\mathcal{T}$ is called the discrete topology.

Continuity

Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be continuous if $f^{-1}(V)$ is open in X for any open set V in Y. In other words:

$$f^{-1}(V) \in \mathcal{T}_X \, \forall V \in \mathcal{T}_Y$$

Exercise

Show that if X has the discrete topology then any function $f : X \to Y$ is continuous, not matter what topology Y has.

Sequences

Let $\{x_n\}$ be some sequence in X. We say that $x \in X$ is a limit of x_n if given any open set U with $x \in U$ we can find an $N > 0$ such that $x_n \in U$ for all $n \geq N$.

Exercise

Let X be some set with the discrete topology. Show that a sequence x_n converges to x if and only if x_n is eventually constant.

Exercise

Let X be some set with the trivial topology. Show that a sequence x_n converges to any $x \in X$. I.e. any sequence converges to all points. At once.

Exercise

Use what you showed about limits in metric spaces to argue that the open sets defining the trivial topology do not come from open sets defined by a metric (unless $X = \{x\}$ or $X = \emptyset$).

Conclude that any metric space is a topological space but the converse is not true.

Hausdorff

A topological space X is said to be Hausdorff (or have the Hausdorff property) if given any pair of points p and q in X you can find open sets U and V with $p \in U$ and $q \in V$ such that $U \cap V = \emptyset$.

Exercise

Show that any metric space has this property.

Exercise

Show that if X consists of at least 2 points then X with the trivial topology will not be Hausdorff.

Exercise

Show that limits are unique in a Hausdorff topological space.

Hint: assume not and let x and y both be limits of a sequence x_n. Pick open sets U and V around x and y respectively with $U \cap V = \emptyset$. Argue why this contradicts the assumption that x_n converges to both x and y.