MAT1100 - Grublegruppe
Extra Problems 7

Jørgen O. Lye

Norms

Closely related to the concept of metrics is the concept of norms, written $\|v\|$ for a vector. For vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ the norm is usually taken to be the length:

$$\|v\| = |v|$$

A norm is supposed to satisfy the following axioms for any vector v and any number α ($\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$):

$$\|v\| \geq 0$$

$$\|v\| = 0 \iff v = 0$$

$$\|\alpha v\| = |\alpha| \, \|v\|$$

$$\|v + u\| \leq \|u\| + \|v\|$$

Given a norm, the reader can hopefully show that

$$d(u, v) = \|u - v\|$$

defines a metric. Note however that not all metrics come from a norm. The discrete metric is one example (why?).

Norms can be defined for functions as well. For instance, we can define

$$\|f\|_p = \left(\int |f|^p \right)^{1/p}$$

This gives rise to the metric discussed previously for the spaces $L^p([0, 1])$.

The goal of this note is to establish the triangle inequality for $1 < p < \infty$, namely that

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Inequalities and integrals

In a sense the most basic inequality we have for integrals is

$$\left| \int f + g \, dx \right| \leq \int |f + g| \, dx$$

This can be thought of as a generalised version of the inequality for sums:

$$\left| \sum_{n=1}^{N} f_n \right| \leq \sum_{n=1}^{N} |f_n|$$

In terms of norms, this inequality says that

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$$

To show the general p case, we need to work a bit.

Young's inequality

Let a, b non-negative numbers and let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

The proof uses that the logarithm is concave. Define $t = 1/p \iff t - 1 = 1/q$, assume $a, b > 0$ and justify the steps used below:

$$\log(ta^p + (1-t)b^q) \geq t \log(a^p) + (1-t)\log(b^q) = \log(a) + \log(b) = \log(ab)$$

Conclude with Young's inequality.
Hint: A function is concave if

$$f(at + (1-t)b) \geq t f(a) + (1-t) f(b)$$

for all $t \in [0, 1]$ and all a, b wherever f is supposed to be defined and concave.

Hölder's inequality

This inequality says that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

where $\frac{1}{p}+\frac{1}{q}=1$. In terms of integrals it says

$$\int_0^1 |fg|\,dx \le \left(\int_0^1 |f|^p\,dx\right)^{1/p}\left(\int_0^1 |g|^q\,dx\right)^{1/q}$$

In the proof, we may assume $0<\|f\|_p<\infty$ and $0<\|g\|_q<\infty$ (why?). By replacing f by $\frac{f}{\|f\|_p}$ and g by $\frac{g}{\|g\|_q}$, we can assume that $\|f\|=\|g\|=1$ (why?).

Use Young's inequality on $a=\frac{|f(t)|}{p}$ and $b=\frac{|g(t)|}{q}$ to conclude that

$$\|fg\|_1 \le \frac{1}{p}+\frac{1}{q}=1=\|f\|_p\,\|g\|_q$$

Minkowski's inequality

Justify the manipulations:

$$\|f+g\|_p^p=\int_0^1 |f{+}g|^p\,dx=\int_0^1 (|f{+}g|)|f{+}g|^{p-1}\,dx \le \int_0^1 (|f|{+}|g|)|f{+}g|^{p-1}\,dx$$

$$=\int_0^1 |f|\cdot|f+g|^{p-1}\,dx+\int_0^1 |g|\cdot|f+g|^{p-1}\,dx$$

Apply Hölder's inequality to these two terms and argue for the following:

$$\left\|(f+g)^{(p-1)}\right\|_q=\left(\int_0^1 |f+g|^{(p-1)q}\,dx\right)^{1/q}=\left(\int_0^1 |f+g|^p\,dx\right)^{1/q}=\|f+g\|_p^{p/q}=\|f+g\|_p^{p-1}$$

Hint: use that $\frac{1}{p}+\frac{1}{q}=1$ several times.

Conclude that

$$\|f+g\|_p^p \le (\|f\|_p+\|g\|_p)\,\|f+g\|_p^{p-1}$$

from which we conclude finally that

$$\|f+g\|_p \le \|f\|_p+\|g\|$$

This is known as Minkowski's inequality and it is the triangle inequality for the L^p-spaces.

Remark

There is nothing inherintly magical about the interval $[0,1]$. Another closed and bounded interval $[a,b]$ would work just as well. If you want these results for an unbounded or open interval, you have to be a bit more careful, as integrals could then quite easily be infinite.