8.4.5 Plan:

8.6.11 c

8.6.15

~~8.3.6~~ ✓

(8.3.9)?

~~8.4.5~~

~~8.5.4~~

(8.5.5)

8.6.1 g

8.6.3

8.6.58

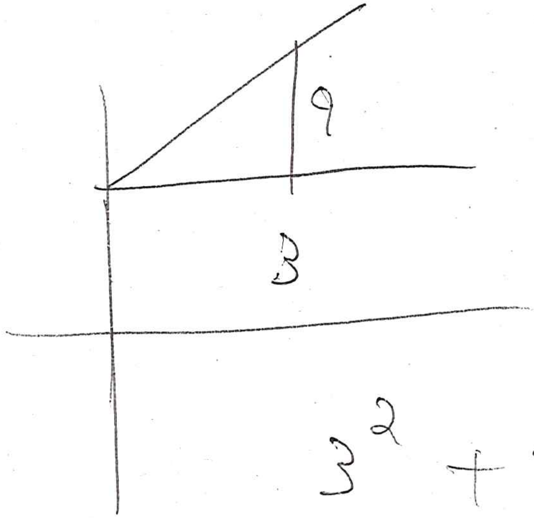
8.6.7 g

~~8.6.11 a~~

~~8.6.11 c~~

8.6.15

5 uker til eksamen (nøyaktig)



$3^2 + 9^2$

Feil eller spørsmål? Send epost til JONASIK @ math. uio. no

8.3.1

a) $$\int_0^{\pi} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi} = -\cos \pi + \cos 0 = -(-1) + 1 = 2$$

b) $$\int_0^{2} 2x^3 \, dx = \left[\frac{2x^4}{4}\right]_0^{2} = \frac{2^4}{2} = 2^3 = 8$$

c) $$\int_0^{1} e^{-x} \, dx = \Big[-e^{-x}\Big]_0^{1} = -e^{-1} + e^{0} = 1 - e^{-1}$$

d) $$\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \Big[\arcsin x\Big]_{-1/2}^{1/2} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

e) $$\int_1^{e} \frac{dx}{x} = \Big[\ln x\Big]_1^{e} = \ln e - \ln 1 = 1$$

f) $$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \Big[\arctan x\Big]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

g) $$\int_{\pi/6}^{\pi/3} \frac{dx}{\sin^2 x} = \Big[-\cot x\Big]_{\pi/6}^{\pi/3} = -\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{1} = \sqrt{3}\left(-\frac{1}{3} + 1\right) = \frac{2\sqrt{3}}{3}$$

h) $$\int_1^{9} x^{3/2} \, dx = \left[\frac{2}{5} x^{5/2}\right]_1^{9} = \frac{2}{5}\left(3^5 - 1\right) = \cancel{\frac{2}{5}(121} \; \frac{2^2 \cdot 11^2}{5}$$

2

8.3.3

a) $\int_0^{2\pi/3} \sin\left(x+\frac{\pi}{3}\right) dx = \left[-\cos\left(x+\frac{\pi}{3}\right)\right]_0^{2\pi/3} = -\cos\frac{\pi}{3} + \cos = \frac{3}{2}$

b) $\int_0^2 e^{3x+2}\, dx = \left[\frac{1}{3}e^{3x+2}\right]_0^2 = \frac{1}{3}\left(e^8 - e^2\right) = \frac{e^2}{3}\left(e^6 - 1\right)$

c) $\int_1^4 \frac{1}{2x+1}\, dx = \left[\frac{1}{2}\ln(2x+1)\right]_1^4 = \frac{1}{2}\left(\ln 9 - \ln 3\right) = \frac{1}{2}\ln\frac{9}{3} = \frac{\ln 3}{2} = \ln\sqrt{3}$

d) $\int_0^{1/2} \frac{dx}{1+4x^2} = \left[\frac{1}{2}\arctan 2x\right]_0^{1/2} = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

e) $\int_0^1 \frac{dx}{\sqrt{9-x^2}} = \frac{1}{3}\int_0^1 \frac{dx}{\sqrt{1-\left(\frac{x}{3}\right)^2}} = \frac{1}{3}\left[3\arcsin\frac{x}{3}\right]_0^1 = \arcsin\frac{1}{3}$

f) $\int_2^3 \left(\sinh 5x + \frac{2}{x-1}\right) dx = \left[\frac{1}{5}\cosh 5x + 2\ln(x-1)\right]_2^3 = \frac{1}{5}(\cosh 15 - \cosh 10) + 2\ln 2$

g) $\int_{-\pi/4}^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{1}{e^{7x}}\right) dx = \left[\tan x \pm \frac{e^{-7x}}{7}\right]_{-\pi/4}^{\pi/4} = 2 + \frac{2}{7}\left(\frac{e^{7\pi/4} - e^{-7\pi/4}}{2}\right)$

$= 1 + \frac{1}{7}\sinh\left(\frac{7\pi}{4}\right)$

3 8.3.6 Anta

a) f er kontinuerlig, g er deriverbar. Vis at hvis $G(x) = \int_0^{g(x)} f(t)\,dt$, så er $G'(x) = f(g(x))\,g'(x)$.

Bevis: La den antideriverte til f være F. Da er

$$\int_a^x f(t)\,dt = F(x) - F(a), \quad \text{og} \quad F'(x) = f(x).$$

Da er $G(x) = \int_0^{g(x)} f(t)\,dt = F(g(x)) - F(a)$, så $G'(x) = D[F(g(x)) - F(a)]$

$$= D[F(g(x))] - 0 = D[F(x)](g(x)) \cdot g'(x) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

b) Plugger inn i a)

(i) $$D\left[\int_0^{\sin x} t e^{-t}\,dt\right] = \sin x\, e^{-\sin x} \cdot \cos x$$

(ii) $$D\left[\int_0^{\sqrt{x}} e^{-t^2}\,dt\right] = e^{-x} \cdot \frac{1}{2\sqrt{x}}$$

$|x| < \frac{\pi}{2}$

(iii) $$D\left[\int_{\sin x}^{0} \frac{dt}{\sqrt{1-t^2}}\right] = D\left[-\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}\right] = -\frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x = -\frac{\cos x}{\cos x} = -1$$

8.3.7 a) Beregn

$$\lim_{x\to 0} \cancel{\int_0^x \frac{e^{-t^2}\,dt}{x}} \quad \frac{\int_0^x e^{-t^2}\,dt}{x},$$

merk at $e^{-t^2} < 1$ for alle t ~~når x~~,

så $\int_0^x e^{-t^2}\,dt \to 0$ når $x \to 0$

$\leq \left|\int_0^x 1\,dt\right| = |x|$

Så vi kan bruke L'Hopital.

$$\overset{L'H}{=} \lim_{x\to 0} \frac{e^{-x^2}}{1} = e^{-0} = 1$$

8.3.9. Anta at f er kontinuerlig.

Vis at det finnes $c \in (a,b)$ slik at $\int_a^b f(x)\,dx = f(c)(b-a)$

Bevis:

La $F(x) = \int_a^x f(t)\,dt$. Da er $F'(x) = f(x)$ og $F(a) = \int_a^a f(t)\,dt = 0$

og $F(b) = \int_a^b f(t)\,dt$

Så middelverdisetningen gir

$$\frac{F(b) - F(a)}{b-a} = F'(c) = f(c) \quad \text{for en } c \in (a,b)$$

altså

$$\cancel{F(b) - 0}$$

$$\frac{\int_a^b f(t)\,dt - 0}{b-a} = f(c) \Rightarrow \int_a^b f(t)\,dt = f(c)(b-a).$$

5

8.4.3 a) $\int \sqrt{\frac{\arcsin x}{1-x^2}}\,dx = \int \sqrt{u}\,du = \cancel{\frac{1}{2\sqrt{u}}} \cancel{= \frac{1}{2\sqrt{\arcsin x}}}$

$= \frac{2}{3}(\arcsin x)^{3/2} + C$

$\left(\begin{array}{l} u = \arcsin x \\ u' = \frac{1}{\sqrt{1-x^2}} \end{array} \right)$

(evt. $\int \sqrt{\frac{\arcsin x}{1-x^2}}\,dx = \int f(g(x))\cdot g'(x)\,dx = F(g(x))$

, der $f(x) = \sqrt{x}$, $F(x) = \cancel{\frac{1}{2\sqrt{x}}}\ \frac{2}{3}x^{3/2}$ og

$g(x) = \arcsin x$, $g'(x) = \frac{1}{\sqrt{1-x^2}}$)

(Husk: $\cos^2 x - \sin^2 x = \cos 2x$)

b) $\int \sin 2x \frac{e^{\cos^2 x}}{e^{\sin^2 x}}\,dx = \int \sin 2x\, e^{\cos^2 x - \sin^2 x}\,dx = \int \sin 2x\, e^{\cos 2x}\,dx$

$= \int \left(-\frac{1}{2}\right)\cdot e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{\cos 2x} + C$

$u = \cos 2x$

$u' = -2\sin 2x \Rightarrow \sin 2x = \left(-\frac{1}{2}u'\right)$

(Indreregel / ytreregel: $\int \sin 2x\, e^{\cos 2x}\,dx = \int \left(-\frac{1}{2}\right)\left(\frac{du}{dx}\right)\cdot e^u \cdot dx = \int e^u$)

siden $u' = \frac{du}{dx}$

c) $\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int 2\frac{u'}{1+u^2}\,\cancel{dx} = \int 2\frac{du}{1+u^2} = 2\int \arctan u$

$= 2\arctan\sqrt{x} + C$

$u = \sqrt{x}$

$u' = \frac{1}{2\sqrt{x}}$

d) $\int \frac{2x-1}{\sqrt{1-x^2}}\,dx = \int \frac{2x}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\left(-\frac{2}{2}u'\right)\cancel{dx}}{\sqrt{u}} - \arcsin x$

$u = 1-x^2$

$u' = -2x$

$\Rightarrow -\frac{2u'}{2} = 2x$

$= -2\int \frac{du}{2\sqrt{u}} - \arcsin x = -2\sqrt{u} - \arcsin x = -2\sqrt{1-x^2} - \arcsin x + C$

6 8.4.5 La $f: (0,\infty) \to \mathbb{R}$

Anta $f(xy) = f(x) + f(y)$ for alle $x, y \in (0,\infty)$

f er deriverbar i $x=1$ med $f'(1) = k$

a) Vis at $f(1) = 0$.

Bevis: $f(1) = f(1\cdot 1) = f(1) + f(1) \Rightarrow f(1) = 0$

b) Vis at $f(x+h) = f(x) + f(1+\frac{h}{x})$, og vis $f'(x) = \frac{k}{x}$.

Bevis: $f(x+h) = f(x(1+\frac{h}{x})) = f(x) + f(1+\frac{h}{x})$.

Siden f er deriverbar i $x=1$ vet vi at

$$\lim_{u\to 0} \frac{f(1+u) - f(1)}{u} = f'(1) = k.$$

Dermed er $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(1+\frac{h}{x})}{h}$

$$= \lim_{h\to 0} \frac{1}{x} \cdot \frac{f(1+h/x)}{h/x} = \lim_{u\to 0} \frac{1}{x} \frac{f(1+u)}{u} = \frac{1}{x} \lim_{u\to 0} \frac{f(1+u)}{u} = \frac{f'(1)}{x} = \frac{k}{x}$$

c) La $g(x) = f(x) - k \ln x$. Da er $g'(x) = f'(x) - \frac{k}{x} = 0$

Da er $g(x) = C$ for et reelt tall C (e.g. Korollar 6.2.4)

men $C = g(1) = f(1) - k \ln 1 = 0 - 0 = 0$ Så $g(x) = 0$. Så

$f(x) = k \ln x$

(evt. analysens fundamentalteorem ...)

7 8.5.4

Vis at $\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i} = \frac{2}{3}$

Bevis: $\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}}$ ~~fint~~

La $f: [0,1] \to \mathbb{R}$ være $f(x) = \sqrt{x}$

$\Pi_n = \{\underset{x_0}{0}, \underset{x_1}{\frac{1}{n}}, \frac{2}{n}, \dots, \frac{n-1}{n}, \underset{x_n}{\frac{n}{n}}\}$, og $u = \{x_1, x_2, \dots x_n\}$, $x_i = \frac{i}{n}$

Da er $R(\Pi_n, u_n) = \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \sum_{i=1}^{n} \underbrace{f(x_i)}_{\sqrt{\frac{i}{n}}} \cdot (\overbrace{x_i - x_{i-1}}^{\left(\frac{i}{n} - \frac{i-1}{n}\right) = \frac{1}{n}})$

og $\lim_{n\to\infty} R(\Pi_n, u_n) = a = \int_0^1 \sqrt{x} = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3}$ ved Teorem 8.5.3

8.5.5 Finn $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \right) = \lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \sqrt{\frac{n}{i}} \right)$

La $f: (0,1] \to \mathbb{R}$ være $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & (1 \geq x > 0) \\ ~~0 \quad (x = 0)~~ \end{cases}$

Bare for å huske bruke 8.5.3, f behøver ikke

$\Pi_n = \{\underset{x_0}{0}, \underset{x_1}{\frac{1}{n}}, \underset{x_2}{\frac{2}{n}}, \dots, \frac{n-1}{n}, \underset{x_n}{\frac{n}{n}}\}$, $u_n = \{x_1, x_2, \dots x_n\}$ $x_i = \frac{i}{n}$

Da er $R(\Pi_n, u_n) = \sum_{i=1}^{n} \underbrace{f(x_i)}_{\sqrt{\frac{n}{i}}} \cdot \underbrace{(x_i - x_{i-1})}_{\frac{1}{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{\frac{1}{i}}$

Så $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \right) = \lim_{n\to\infty} R(\Pi_n, u_n) \overset{?}{=} \int_0^1 \frac{1}{\sqrt{x}} \overset{\text{~~merk at~~}}{=} \left[2\sqrt{x} \right]_0^1 = 2$

(merk at f ikke er definert på $[0,1]$, og $\left(\frac{1}{\sqrt{x}}\right)$ er ikke begrenset på $(0,1]$, så vi må forklare hvorfor $\lim_{n\to\infty} R(\Pi_n, u_n) = \int_0^1 \frac{1}{\sqrt{x}}$ ~~dx~~, Teorem 8.5.3 kan ikke anvendes direkte)

Vi vet $R(\Pi_n, u_n) \leq \int_{1/n}^{1} \frac{1}{\sqrt{x}}$, siden $R(\Pi_n, u_n)$ er et nedre integral for $\frac{1}{\sqrt{x}}$ på $(\frac{1}{n}, 1)$

$= \int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{x}}$,

La $\Pi_n' = \{\underset{x_0'}{\frac{1}{n}}, \underset{x_1'}{\frac{2}{n}}, \dots, \underset{x_n'}{\frac{n}{n}}, \frac{n+1}{n}\}$, $u_n' = \{x_0', x_1', \dots x_n'\}$, $x_i' = \frac{i+1}{n}$

$\Rightarrow R(\Pi_n, u_n) = R(\Pi_n', u_n')$ ~~er et øvre integral for~~ $\geq \int_{1/n}^{1+\frac{1}{n}} \frac{1}{\sqrt{x}}$

så $\Rightarrow \int_{1/n}^{1+\frac{1}{n}} \frac{1}{\sqrt{x}} \leq R(\Pi_n, u_n) \leq \int_{1/n}^{1} \frac{1}{\sqrt{x}} \Rightarrow \lim_{n\to\infty} R(\Pi_n, u_n) = 2$

7.5
8.5.5 Finn $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}}\right)$ (Se neste side også)

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}}\right) = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} \sqrt{\frac{n}{i}}\right)$$

La $f(x) = \frac{1}{\sqrt{x}}$, $f: (0,1] \to \mathbb{R}$

La $\Pi_n = \{\frac{1}{n}, \frac{2}{n}, \dots \frac{n+1}{n}\}$, $x_i = \frac{i+1}{n}$, være
$x_0, x_1, \dots x_n$

en partisjon av $[\frac{1}{n}, \frac{n+1}{n}]$

Da er $\emptyset(\Pi_n) = \sum_{i=1} M_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{1}{n}$, siden f er monoton

$= \frac{1}{n}\sum_{i=1}^{n} \sqrt{\frac{n}{i}}$, og dermed er $\int_{1/n}^{\frac{n+1}{n}} \frac{dx}{\sqrt{x}} \leq \emptyset(\Pi_n) = \frac{1}{n}\sum_{i=1}^{n} \sqrt{\frac{n}{i}}$. (1)

La $P_n = \{\frac{1}{2n}, \frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}\}$ være en partisjon av
$x_0, x_1, x_2, \dots$

$[\frac{1}{2n}, 1]$. Da er $N(\Pi_n) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$

$$= \sqrt{\frac{n}{1}} \cdot \left(\frac{1}{n} - \frac{1}{2n}\right) + \sum_{i=2}^{n} \sqrt{\frac{n}{i}} \underbrace{(x_i - x_{i-1})}_{1/n} = \sum_{i=1}^{n} \sqrt{\frac{n}{i}} \cdot \frac{1}{n} - \sqrt{\frac{n}{1}} \cdot \frac{1}{2n}$$

og dermed $\int_{1/2n}^{1} \frac{dx}{\sqrt{x}} \geq N(\Pi_n) = \sum_{i=1}^{n} \sqrt{\frac{n}{i}} \cdot \frac{1}{n} - \frac{1}{2\sqrt{n}}$,

eller ekvivalent, $\int_{1/2n}^{1} \frac{dx}{\sqrt{x}} + \frac{1}{2\sqrt{n}} \geq \frac{1}{n}\sum_{i=1}^{n} \sqrt{\frac{n}{i}}$. (2)

Altså gir (1) + (2)

$$\int_{1/n}^{\frac{n+1}{n}} \frac{dx}{\sqrt{x}} \leq \frac{1}{n}\sum_{i=1}^{n} \sqrt{\frac{n}{i}} \leq \int_{1/2n}^{1} \frac{dx}{\sqrt{x}} + \frac{1}{2\sqrt{n}},$$ så tar vi grenser over n og får

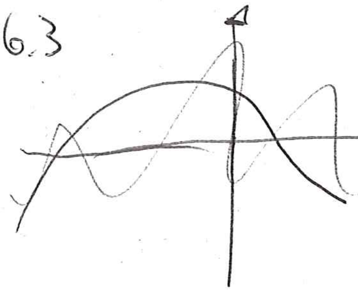
$$2 = \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{n\to\infty} \int_{1/n}^{\frac{n+1}{n}} \frac{dx}{\sqrt{x}} \leq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sqrt{\frac{n}{i}} \leq \lim_{n\to\infty}\left(\int_{1/2n}^{1} \frac{dx}{\sqrt{x}} + \frac{1}{2\sqrt{n}}\right) = \int_0^1 \frac{dx}{\sqrt{x}} = 2$$

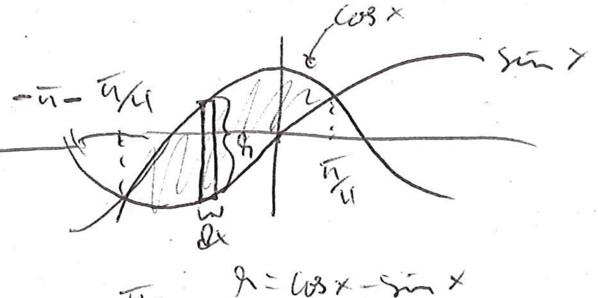
7.25

8.5.5 er nok litt vanskeligere enn den burde være. Problemet er at vi gjenkjenner $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}}\right)$ som en Riemannsum for $\frac{1}{\sqrt{x}}$ på $(0,1)$. Men $(0,1)$ er ikke et lukket intervall, og $\frac{1}{\sqrt{x}}$ er ikke begrenset på $(0,1)$. Så vi kan ikke anvende Teorem 8.5.3. Faktisk har vi ikke lært å integrere ubegrensede funksjoner ennå (se definisjon 8.5.2 og 8.2.1). ~~Vi må~~ Integralet $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ blir først definert i 9.5.6.

Hvis man brukte en annen definisjon av hva det vil si å være Riemann integrerbar enn den som er gitt i 8.5.2, og hadde en mer generell variant av 8.5.3, ville oppgaven være betydelig enklere. Vi kunne valgt partisjonen $\Pi = \{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}\}$, og utvalget $\{\frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}\}$ og argumentert akkurat som i oppgave 8.5.4

8.6.3





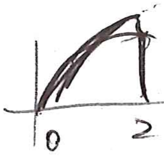
Finn arealet. Så $A = \int_{-\pi-\pi/4}^{\pi/4} (\cos x - \sin x)\, dx$

(Eventuelt formel på side 439 første avsnitt i 8.6)

$$= \Big[\sin x + \cos x\Big]_{-5\pi/4}^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

8.6.5 a) Finn volumet til omdreiningslegemet når grafen dreies om x-aksen



$y = \sqrt{x} \quad x = 0, 2$

$$V = \pi\int_0^2 (\sqrt{x})^2\, dx = \pi\left[\frac{x^2}{2}\right]_0^2 = 2\pi$$

b) $y = \frac{1}{\sqrt{1+x^2}}$

$$V = \pi\int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \pi\Big[\arctan x\Big]_0^1 = \pi \cdot \frac{\pi}{4} = \left(\frac{\pi}{2}\right)^2$$

d)

$$V = \pi\int_{\pi/6}^{\pi/3} \left(\frac{1}{\sin x}\right)^2 = \pi\Big[-\cot x\Big]_{\pi/6}^{\pi/3} = \pi\left(-\frac{1}{\sqrt{3}} + \sqrt{3}\right) = \frac{2\pi\sqrt{3}}{3}$$

$y = \frac{1}{\sin x}$

9 8.6.7 a) $y = x^2$ $x = 0$, $x = 3$ Volum når vi dreier om y-aksen

$$V = 2\pi \int_a^b x f(x)\, dx = 2\pi \int_0^3 x^3 = 2\pi \left[\frac{x^4}{4} \right] = \frac{\pi}{2} 3^4 = \frac{\pi 81}{2}$$

c) $y = \frac{1}{1+x^2}$ på $[0, 2]$

$$2\pi \int_0^2 \frac{x}{1+x^2}\, dx = \pi \int_0^2 \frac{2x}{1+x^2}\, dx = \pi \left[\ln(1+x^2) \right]_0^2 = \pi \ln 5$$

g) $y = \frac{1}{1+x^4}$ på $[0, 1]$

$$2\pi \int_0^1 \frac{x}{1+x^4}\, dx = \cancel{2\pi \left[\ln(1+x^4) \right]_0^1} = \pi \int_0^1 \frac{2x}{1+(x^2)^2}\, dx$$

$$= \pi \int_0^1 \frac{du}{1+u^2} = \pi \left[\arctan u \right]_0^1 = \pi \cdot \frac{\pi}{4} = \frac{\pi^2}{4} = \left(\frac{\pi}{2}\right)^2$$

$u = x^2$

$du' = 2x$

10

buelengde

8.6.14 a) $y = 3x + 4$ på $[0, 3]$

$$L = \int_0^3 \sqrt{1 + f'(x)^2}\, dx = \int_0^3 \sqrt{1 + 3^2}\, dx = \sqrt{10} \cdot 3$$

c) $y = \frac{x^2}{2} - \frac{1}{4} \ln x = f(x)$ på $(1, e)$, $f'(x) = x - \frac{1}{4x}$

$$L = \int_1^e \sqrt{1 + f'(x)^2}\, dx = \int_1^e \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\, dx$$

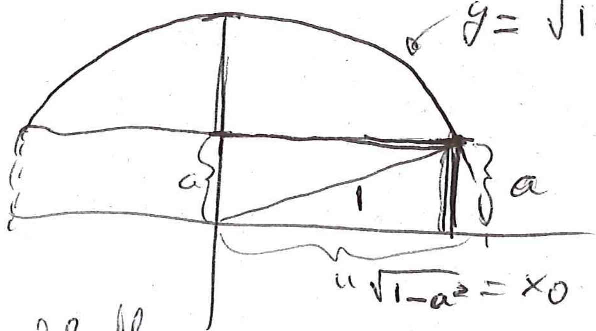
$$= \int_1^e \sqrt{1 + x^2 - \frac{1}{2} + \left(\frac{1}{4x}\right)^2}\, dx = \int_1^e \sqrt{x^2 + \frac{1}{2} + \left(\frac{1}{4x}\right)^2}\, dx = \int_1^e \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx$$

$$= \int_1^e \left(x + \frac{1}{4x}\right) dx = \left[\frac{1}{2}x^2 + \frac{1}{4} \ln x\right]_1^e = \frac{e^2}{2} + \frac{1}{4} - \frac{1}{2} = \frac{e^2}{2} - \frac{1}{4}$$

8.6.15 Vi borrer et sylindrisk hull med radius $a<1$ i en kule med radius 1. ~~gjennom kulens~~ ~~Hullet går gjennom~~ sylinderens akse går gjennom sentrum av kulen.

Beregn volumet.

Vi tar et tverrsnitt:



Kulen med hull er omdreiningslegemet til denne figuren

Vi beregner volumet ved å finne volumet V_1 ~~to~~ omdreiningslegemet av $y=\sqrt{1-x^2}$ på $[-x_0, x_0]$ og volumet V_2 av omd.—||— av $y=a$ på $[-x_0, x_0]$

$$V_1 = \pi\int_{-x_0}^{x_0}\left(\sqrt{1-x^2}\right)^2 dx = \pi\int_{-x_0}^{x_0}(1-x^2)\,dx = \pi\left[x-\frac{x^3}{3}\right]_{-x_0}^{x_0} = \frac{2\pi x_0}{3}(3-x_0^2)$$

$$V_2 = \pi\int_{-x_0}^{x_0} a^2 = 2\pi\cdot a^2\cdot x_0$$

$$V = V_1 - V_2 = \frac{2\pi x_0}{3}(3-x_0^2) - 2\pi a^2\cdot x_0 = \frac{2\pi x_0}{3}(3-x_0^2-3a^2)$$

$$= \frac{2\pi\sqrt{1-a^2}}{3}(3-1+a^2-3a^2) = \frac{2\pi\sqrt{1-a^2}}{3}(2-2a^2) = \frac{4\pi}{3}(1-a^2)^{3/2}$$

(merk at dette gir riktig svar når $a=0$ og $a=1$)