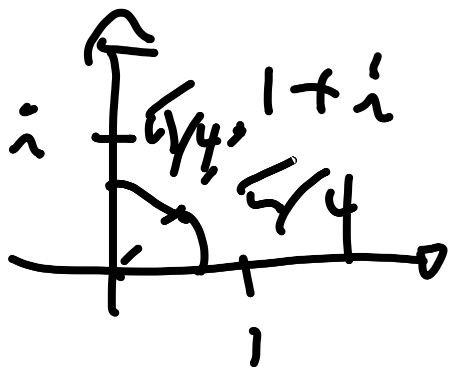
3.3.8 Regn ut $(1+i)^{804} = ?$

Husk: Komplekse tall kan skrives på polar form:

$1+i = re^{i\theta}$ for r et reelt tall og $\theta \in [0, 2\pi)$



$$1+i = \sqrt{2}\, e^{i\pi/4}$$

$$(1+i)^{804} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{804}$$

$$= \left(2^{1/2} e^{i\pi/4}\right)^{804} = 2^{804/2} e^{i\frac{\pi\cdot 804}{4}}$$

$$= 2^{402}\cdot e^{i\pi 201} = 2^{402}\cdot \underbrace{e^{i\pi\cdot 200}}_{=1}\cdot \underbrace{e^{i\cdot\pi}}_{=-1}$$

$$= 2^{402}\cdot 1\cdot(-1) = \underline{\underline{-2^{402}}}$$

$(\sqrt{3}-i)^{173} = ?$ .....

$$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$= \overline{2e^{i\pi/6}} = 2e^{-i\pi/6}$$

osv. ....

3.3.12 b) Vis at $\sum_{k=0}^{n} e^{ik\theta} = \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}$

La $z = e^{i\theta}$ (θ er ikke et heltalls multiplum av 2π)

Da

$$\sum_{k=0}^{n} \underbrace{\left(e^{i\theta}\right)^k}_{= e^{ik\theta}} = \frac{\overbrace{\left(e^{i\theta}\right)^{n+1}}^{= e^{i(n+1)\theta}} - 1}{e^{i\theta}-1}$$

$\blacksquare$

c) Vis at

$$\sum_{k=0}^{n} e^{ik\theta} = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

$$= e^{i\frac{n\theta}{2}} \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

$$\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} = \frac{e^{i(\frac{n}{2}+\frac{n}{2}+1)\theta} - e^{(i\frac{n}{2}-i\frac{n}{2})\theta}}{e^{i\theta} - 1}$$

$$= e^{i\frac{n}{2}\theta} \frac{\left(e^{i(\frac{n}{2}+1)\theta} - e^{-i\frac{n}{2}\theta}\right)}{e^{i\theta} - 1}$$

gang med $e^{-i\theta/2}$ over og under brøken

$$= e^{i\frac{n}{2}\theta} \frac{\left(e^{i(\frac{n}{2}+1)\theta} \cdot e^{-i\theta/2} - e^{-i\frac{n}{2}\theta} \cdot e^{-i\theta/2}\right)}{e^{i\theta} \cdot e^{-i\theta/2} - e^{-i\theta/2}}$$

$$= e^{in\theta/2} \frac{\left(e^{i(\frac{n}{2}+\frac{1}{2})\theta} - e^{-i(\frac{n+1}{2})\theta}\right)}{e^{i\theta/2} - e^{-i\theta/2}}$$

$$= e^{in\theta/2} \frac{\cancel{2i} \operatorname{Im} e^{i(\frac{n+1}{2})\theta}}{\cancel{2i} \operatorname{Im} e^{i\theta/2}}$$

Husk $e^{i\theta} = \cos\theta + i\sin\theta$

$\operatorname{Im} e^{i\theta} = \sin\theta$

$$= e^{in\theta/2} \cdot \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\theta/2}$$

$\boxtimes$

Vis at $\sum_{k=0}^{n} \cos k\theta$

$$= \frac{\cos\left(\frac{n\theta}{2}\right)\sin\left(\frac{n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}.$$

Bevis: Vi vet

$$\sum_{k=0}^{n} e^{ik\theta} = e^{i\frac{n\theta}{2}} \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

$$\parallel$$

$$\sum_{k=0}^{n} (\cos k\theta + i\sin k\theta)$$

$$\parallel$$

$$\cos\frac{n\theta}{2} \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\theta/2} + i\left(\sin\frac{n\theta}{2} \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\theta/2}\right)$$

Så siden real delene til venstre og høyre side er like så får vi likheten vi ønsket.

3.5.7 Vis at $P(z) = z^3+2z^2-3z+20$

har $r = 1-2i$ som en rot

$$r = 1-2i$$
$$r^2 = (1-2i)^2 = 1-4i-4 = -3-4i$$
$$r^3 = (-3-4i)(1-2i) = -3-4i+6i-8$$
$$= -11+2i$$

$$P(r) = -11+2i+2(-3-4i)-3(1-2i)+20$$
$$= -11+2i-6-8i-3+6i+20$$
$$= 0 \qquad 0 \qquad = 0$$

Vi vet at $1-2i$ er en rot

Siden P er et reelt polynom

vil $\overline{1-2i} = 1+2i$ også

være en rot.

Så vi vet at $(z-1+2i)(z-1-2i)$

$= (z-1)^2+4 = z^2-2z+5$

Deler $P(z)$, så vi gjør

polynomdivisjon:

$$(z^3+2z^2-3z+20):(z^2-2z+5) = z+4$$
$$-z^3+2z^2-5z$$
$$4z^2-8z+20$$

så den reelle roten er -4

Så $P(z) = (z^2-2z+5)(z+4)$

er den reelle faktoriseringen

Siden (z^2-2z+5) har røtter

med ikke-null imaginær del

Kompleks faktorisering er

da:

$$(z-1+2i)(z-1-2i)(z+4).$$

3.5.13 a) Finn de komplekse kvadratrøtter til $-1+i\sqrt{3}$

Da er $-\frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}$

b) Løs $P(z) = z^4 + z^2 + 1 = 0$

Triks: La $u = z^2$

Da er $P(u) = u^2 + u + 1$

Så vi vil faktorisere $P(u)$

$P(u)$ har røtter: $\frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

Så $z^2 = u = \frac{-1 \pm i\sqrt{3}}{2}$, så vi må

finne kvadratrøtten til $\frac{-1+i\sqrt{3}}{2}$

& til $\frac{-1-i\sqrt{3}}{2}$

Røttene til $\frac{-1+i\sqrt{3}}{2}$ fant vi over

de var $e^{i\pi/3}$ & $-e^{i\pi/3}$

Så siden $P(z)$ er et reelt polynom må $\overline{e^{i\pi/3}} = e^{-i\pi/3}$

være en rot &

$-e^{i\pi/3}$ også være

$-e^{-i\pi/3}$ en rot.

Så $P(z)$ kan faktoriseres som følger

$$P(z) = (z - e^{i\pi/3})(z + e^{i\pi/3})$$
$$(z - e^{-i\pi/3})(z + e^{-i\pi/3})$$

Reell faktorisering

multipliser og

Nyttig å huske:

$$e^{i\pi/2} = i$$

$$e^{i\pi} = -1$$

$$e^{i2\pi} = e^{i\cdot 0} = 1$$

$$e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$\overline{e^{i\theta}} = e^{-i\theta} =$$

Hvis $e^{i\theta} = a + ib$

Så er $e^{-i\theta} = a - ib$

$$e^{i\pi 201}$$

$$\|$$

$$e^{i\pi 200} \cdot e^{i\pi}$$

3.3.9 V is at $\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n$

$$= \frac{1+i\tan n\theta}{1-i\tan n\theta}$$

Bevis: Husk: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\frac{1+i\tan x}{1-i\tan x} = \frac{1+i\frac{\sin x}{\cos x}}{1-i\frac{\sin x}{\cos x}} \begin{matrix} 1\cdot\cos x \\ \\ 1\cdot\cos x \end{matrix} = \frac{\cos x + i\sin x}{\cos x - i\sin x}$$

$$= \frac{?}{?} = \frac{e^{ix}}{e^{-ix}} \left(= e^{i2x}\right)$$

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \left(\frac{e^{i\theta}}{e^{-i\theta}}\right)^n$$

$(x=\theta)$

$$= \frac{e^{in\theta}}{e^{-in\theta}} = \frac{1+i\tan n\theta}{1-i\tan n\theta}$$

$(x=n\theta)$

3.4.3 Finn tredjerøttene til i.

Så vi må løse $z^3 = i$.

Vet $i = e^{i\frac{\pi}{2} + i2\pi k}$, for k et heltall

så $i^{1/3} = e^{\left(i\frac{\pi}{2} + i2\pi k\right)/3}$

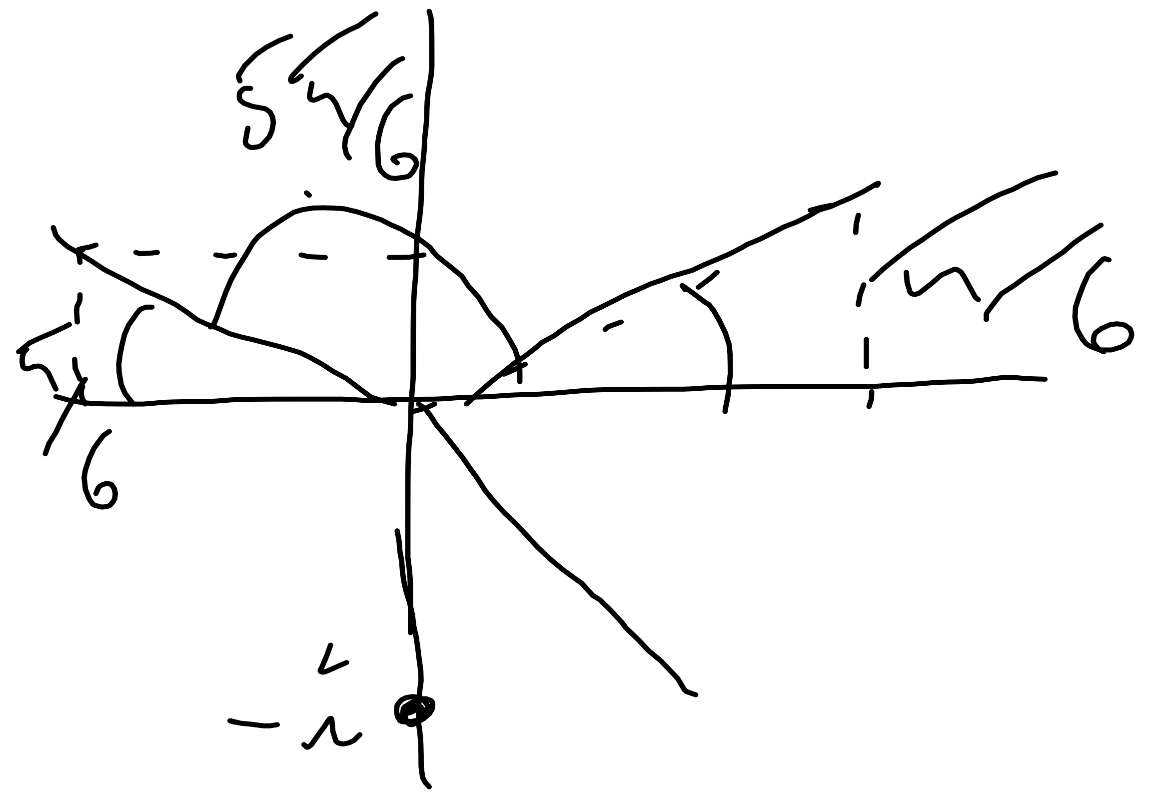
$= e^{i\frac{\pi}{6} + i\frac{2\pi k}{3}}$

$k = 0$: $e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$k = 1$: $e^{i\left(\frac{\pi}{6} + \frac{4\pi}{6}\right)} = e^{i5\pi/6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$k = 2$: $e^{i\left(\frac{\pi}{6} + \frac{8\pi}{6}\right)} = e^{i\left(\frac{9\pi}{6}\right)} = -i$

$\frac{9\pi}{6} = \frac{3\pi}{2}$



Så tredjerøttene til i er

$\frac{\sqrt{3}}{2} + \frac{i}{2}$, $-\frac{\sqrt{3}}{2} + \frac{i}{2}$,

$-i$

3.4.9a) Lös $x^2+2x+4=0$

$$ax^2+bx+c$$

$$r_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4-16}}{2}$$

$$= \frac{-2\pm\sqrt{-12}}{2} = \frac{-2\pm\sqrt{(-1)\cdot 3\cdot 4}}{2}$$

$$= \frac{-2\pm 2i\sqrt{3}}{2} = \underline{\underline{-1\pm i\sqrt{3}}}$$

er lösningen.

3.4.11 c) Lös: $z^2+2z-i\sqrt{3}=0$

$\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$ $az^2+bz+c=0$

$\frac{-2\pm\sqrt{4+4i\sqrt{3}}}{2}=\frac{-2\pm\sqrt{4(1+i\sqrt{3})}}{2}$

$=\frac{-2\pm 2\sqrt{1+i\sqrt{3}}}{2}=-1\pm\sqrt{1+i\sqrt{3}}$

$\sqrt{1+i\sqrt{3}}$. Alltså: $1+i\sqrt{3}=2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

$=2e^{i\pi/3}$

så roten av $\sqrt{2e^{i\pi/3}}=\sqrt{2}e^{i\pi/6}$

$(-\sqrt{2}e^{i\pi/6})$

$=\sqrt{2}e^{i\pi/6}=\sqrt{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$

$=\frac{\sqrt{6}}{2}+i\frac{\sqrt{2}}{2}$

Så lösningarna är

$$-1\pm\sqrt{1+i\sqrt{3}}=\underline{\underline{-1\pm\left(\frac{\sqrt{6}}{2}+\frac{i\sqrt{2}}{2}\right)}}$$

3.4.15 Finn løsningene til

$$z^3 + iz^2 + z = 0$$

$$\Vert$$

$$z(\underbrace{z^2 + iz + 1}_{az^2 + bz + c})$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-i \pm \sqrt{-1 - 4}}{2}$$

$$= \frac{-i \pm \sqrt{-5}}{2} = \frac{-i \pm i\sqrt{5}}{2}$$

Så $z^2 + iz + 1 = 0$ har løsningene $-\frac{i}{2}(1 \mp \sqrt{5})$

Så $z^3 + iz^2 + z = 0$ har komplekse løsningene $z = 0 \vee -\frac{i}{2}(1 - \sqrt{5})$

$\vee -\frac{i}{2}(1 + \sqrt{5})$

3.3.12 a) Vis at $\displaystyle\sum_{k=0}^{n} z^k = \frac{z^{n+1}-1}{z-1}$ (når $z \neq 1$)

Bevis: Samme for reelle tall

La $S = \displaystyle\sum_{k=0}^{n} z^k = 1 + z + z^2 + \dots + z^n$

$S(z-1) = \quad = (z-1)\cdot 1 + (z-1)\cdot z + (z-1)\cdot z^2 + \dots$

$\ \cancel{z}-1 + \cancel{z^2}-\cancel{z} + \cancel{z^3}-\cancel{z^2} \dots$

$+ (z-1)z^n$

$\dots \; z^{n+1} - \cancel{z^n} = z^{n+1} - 1$

$S(z-1) = z^{n+1} - 1$

$$S = \frac{z^{n+1}-1}{z-1}$$

Litt annen vri:

$$\left(\sum_{k=0}^{n} z^k\right)(z-1) = \sum_{k=0}^{n} z^{k+1} - \sum_{k=0}^{n} z^k$$

$$= z^{n+1} + \cancel{\sum_{k=1}^{n} z^k} - \left(\cancel{\sum_{k=1}^{n} z^k} - 1\right)$$

$$= z^{n+1} - 1$$

$\blacksquare$