The definition tells us that given two elements a and b in a totally ordered set, either $a \leq b$ or $b \leq a$. For elements in a partially ordered set that is not totally ordered, there is a third possibility. **What is it?**

4.2.4 EXERCISE

Let A be a set. Show that $\mathcal{P}(A)$ need not be totally ordered under the relation $\subseteq$. □

4.2.5 DEFINITION

Let A be a partially ordered set under $\leq$. A is said to obey the **law of tricotomy** if for every a and $b \in A$ exactly one of the following is true:

i. $a < b$. **ii.** $a = b$. **iii.** $b < a$.

4.2.6 THEOREM

A partially ordered set A is totally ordered iff it obeys the law of trichotomy. □

Let $(A, \leq)$ be any partially ordered set and B any subset of A. Then B *inherits* the partial order from A, in the sense that if we take two elements x and y of B, we can sensibly ask whether $x \leq y$, $y \leq x$, or x and y are unrelated. (You should verify that the relation on B obtained in this way is a partial order.) In other words, we can easily view any subset of a partially ordered set as a partially ordered set **in its own right.**

4.2.7 EXERCISE

It is easy to see that $\mathcal{P}(\mathbb{N})$ is not totally ordered, but it has subsets that are totally ordered. Give an example of an infinite totally ordered subset of $\mathcal{P}(\mathbb{N})$ (under the order $\subseteq$). □

When we are dealing with a partial order on a set with only a few elements, we can often make sense of the situation with some simple diagrams called **lattice diagrams**.[3] (Lattice diagrams can also be useful for picturing a small section of a larger ordered set.)

[3]One reader pointed out that this is a misnomer. Lattices are partially ordered sets in which every pair of elements has a least upper bound and a greatest lower bound. (See page 76 for definitions.) We use the diagrams to represent partially ordered sets that are not lattices. The misusage is common, so I let it stand, but the point is well taken.

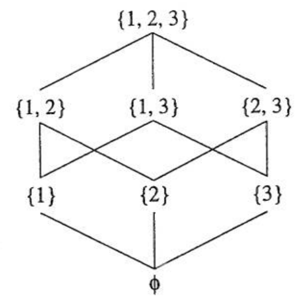


Figure 4.4 Lattice diagram for $\mathcal{P}(A)$

Consider, for instance, the power set of $A = \{1, 2, 3\}$ under set inclusion. We can draw a simple diagram that shows the order relation on this set. This is illustrated in Figure 4.4.

One element of the set is smaller than another if and only if an upward path can be found that connects them. (That path may traverse other elements in between.) Since we always move up along the diagram, elements that are on the same level cannot be related to one another, nor can an element above be less than one that sits on a level below it. Naturally, each element is related to itself as in all partial orders. This is not shown explicitly in the diagram.

4.2.8 EXERCISE

When we say that "one element of the set is smaller than another only if an upward path can be found that connects them" (even if it traverses other elements in between), what property of partial orderings are we relying on? □

Remark. The drawing of lattice diagrams is a pretty intuitive enterprise, so I will mostly leave it to your intuition and will not try to rigorously write out rules for constructing them. However, there is one rule that is worth mentioning explicitly. An element always appears on the *lowest possible level*. That is, an element must always be larger than at least one element on the level immediately below it. If this is not the case, we can and will draw it on a lower level.

4.2.9 EXERCISE

Show that the partial orderings on $\{a, b, c, d, e, f\}$ depicted in Figure 4.5 are the same. (Remember, partial orderings are sets of ordered pairs. Two partial orders are same if those sets are equal.) Notice, however, that only one of the three lattice diagrams is "legal." Which one is it? □

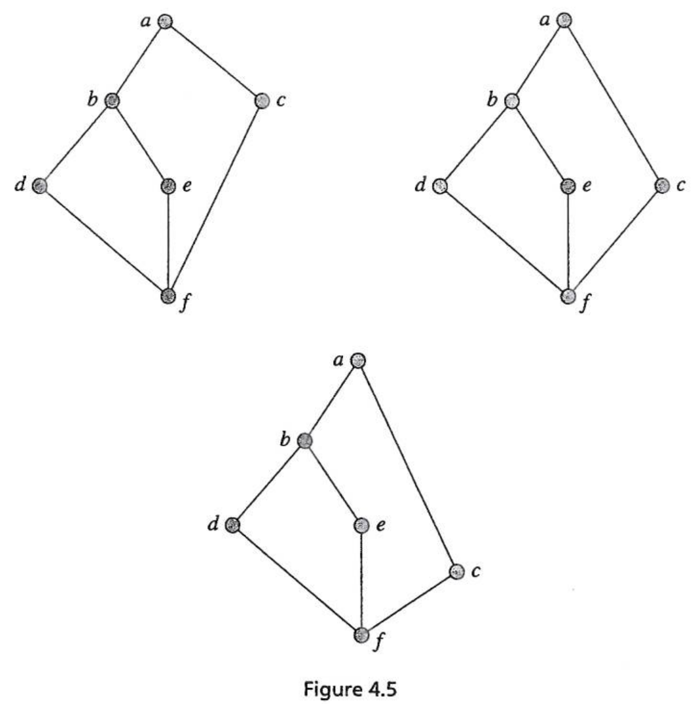


Figure 4.5

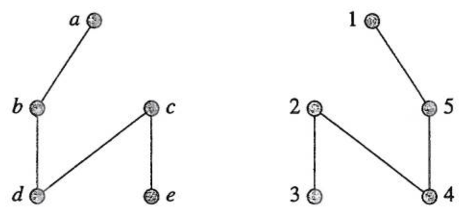


Figure 4.6 Isomorphic partial orders

Consider the diagrams shown in Figure 4.5. Suppose that we had not labeled the ements. Then we would have been observing only the order structure of the set and othing else. There is a sense in which the two lattice diagrams shown in Figure 4.6 e the *same* mathematical structure. The difference between them amounts only to a

Figure 4.7 Partial orders on two elements

"relabeling" of the points. As long as we are interested only in the order structure, such relabeling is not important.[4]

Two mathematical structures that are the same up to a relabeling of the elements involved are called **isomorphic** structures. The partial orderings shown in Figure 4.6 are, therefore, isomorphic partial orders. We will have more to say about isomorphic partial orders when we talk about functions. For now, we will make do with an intuitive idea of what isomorphism means.

The etymology of the word "isomorphic" is enlightening. It comes from the two greek words *isos* meaning *same* and *morphos* meaning *shape*. Things that are isomorphic have the same mathematical "shape" or structure.

4.2.10 EXAMPLE

Suppose we have sets with two, three, and four elements. How might these sets be partially ordered (up to a relabeling of the elements—*up to isomorphism*)? We can use lattice diagrams to find all possible situations.

1. *Partial orders on sets with two elements.* For the set with two elements, we have only two possibilities: Either one element is smaller than the other or they are not related. This yields only two possible partial orders (which we display in Figure 4.7).
2. *Partial orders on sets with three elements.* Draw lattice diagrams for all possible partial orders on a set with three elements. There are five of them. (*Hint:* If you have trouble, you might peek at the discussion given below about partial orders on a set with four elements and then try again.)
3. *Partial orders on sets with four elements.* As the number of elements gets larger, the number of possible partial orders increases rapidly, so if we have any hope of getting all possibilities, we have to be quite systematic about the way we go about classifying them. One possible approach is to think about the number of "levels" there will be in a lattice diagram. If we have four elements, we can have our elements on one, two, three, or four levels. (If all the elements are on the

[4] You have seen this before. The functions

$$f(x) = x^4 - 1 \quad \text{and} \quad g(y) = y^4 - 1$$

are, in fact, the same mathematical object.

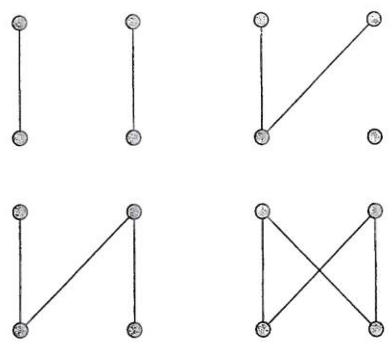


Figure 4.8 Partial orders on a set with four elements in which there are two elements on each level

same level, we have a totally unordered set. If the elements lie on four levels, we have a totally ordered set.)

Let us consider partial orders in which there are only two levels. Here there are three possibilities:

- There are two elements on each level.
- There are three elements on the first (lowest) level and one on the second level.
- There is a single element on the first level and there are three on the second level.

Consider the case in which there are two elements on each level. Remember that the rules for drawing lattice diagrams require that each element on the second level be "tied" to at least one element on the first level. This gives us four possibilities. Each element on the first level is tied to a different element on the second level. One of the elements on the first level is tied to both elements on the second level, and the other element is unrelated to the elements on the second level. One of the elements on the first level is tied to each element on the second level and the other element is tied to only one of them. The final possibility is that each element on the first level is tied to each element on the second level. This line of reasoning yields the lattice diagrams shown in Figure 4.8.

Now you should attempt to finish the classification of partial orders on sets with four elements. (There are 16 of them!)

4. Draw lattice diagrams for at least ten distinct partial orders on sets with five elements. Can you get all of them? (There are 55 in all.) ■

Though lattice diagrams can be very useful for describing partial orderings on small ets, they are of limited usefulness when the sets we are dealing with are large or infinite. Analysis of partial orderings on larger sets therefore requires that we be able to describe

ordered sets and totally unordered sets. These are what you might call *global properties* of a partial order—that is, properties that describe the partial order as a whole. Some elements have a special status with respect to all the other elements in the partially ordered set.

4.2.11 DEFINITION

Let A be a partially ordered set. Let $x \in A$.

1. The element x is called a **maximal element** for A if there exists no $y \in A$ such that $y > x$. Similarly, x is a **minimal element** for A if there does not exist $y \in A$ such that $y < x$.
2. The element x is the **greatest element** of A if $x \geq y$ for all $y \in A$. Similarly, x is the **least element** of A if $x \leq y$ for all y in A.

At first glance, you might think that "greatest element" and "maximal element" mean the same thing, but, in fact, they are different. One reason why you might confuse these two is that you are too used to thinking about totally ordered sets!

4.2.12 EXERCISE

Look at the lattice diagrams in Figure 4.21 on page 98. For each determine whether the partially ordered set has any minimal elements, maximal elements, a greatest element, a least element. In each case, list all you find. □

4.2.13 EXERCISE

1. Give examples that illustrate the difference between a maximal element and the greatest element of a partially ordered set A. Drawing lattice diagrams is a good way to do this.
2. Give examples to show that a given partially ordered set can have more than one minimal element or none at all. □

4.2.14 THEOREM

Let $\mathcal{T}$ be a totally ordered set and fix $x \in \mathcal{T}$. Then x is a maximal element of $\mathcal{T}$ if and only if x is the greatest element of $\mathcal{T}$. (That is, in a totally ordered set, the phrases "maximal element" and "greatest element" mean the same thing!) □

Notice that we have been talking about *the* greatest element of A, as though there were only one. The following theorem justifies this usage.

4.2.15 THEOREM

Let A be a partially ordered set. Prove that if A has a greatest element, then that greatest element is unique. (*Hint:* This is a uniqueness theorem. Turn back to page 30 and remind

Remark. Theorems 4.2.14 and 4.2.15 dealt with greatest elements and maximal elements. Analogous statements can be made about least elements and minimal elements. Formulate the appropriate statements, then review the arguments you gave for maximal and greatest elements to see that they can be easily modified to give the analogous results for least and minimal elements.

In addition to global properties, we have *local properties,* that is, properties that refer only to a specific section (subset) of the partially ordered set—they describe only how the partial order behaves "locally" and have nothing to say about the larger picture in the partial order.

In general, when we define an object, we cannot assume it is unique just because we wish it so. We are defining the object by means of a property or set of properties. Any objects satisfying those properties will then satisfy the definition. Furthermore, we cannot just add the condition that it must be unique to the definition; if we did, the definition would still not distinguish between the various elements that satisfy the other properties. Thus when we make a definition, we must prove the uniqueness of the object (if indeed it is unique), or accept that there may be more than one object that satisfies the definition.

4.2.16 DEFINITION

Let A be a partially ordered set. An element x of A is said to be an **immediate successor** of $y \in A$ if $y < x$ and there does not exist an element $z \in A$ such that $y < z < x$.

Likewise, $x \in A$ is said to be an **immediate predecessor** of $y \in A$ if . . . (You should complete this definition for yourself.)

4.2.17 EXERCISE

Show by giving an example that immediate successors and immediate predecessors are not necessarily unique. □

4.2.18 THEOREM

In a totally ordered set, immediate successors and immediate predecessors (when they exist) are unique. □

4.2.19 DEFINITION

Let A be a partially ordered set. Let K be a nonempty subset of A. Let $x \in A$.

1. x is an **upper bound** for K if $x \geq y$ for all $y \in K$. If such an element exists, we say that K is **bounded above**.
2. x is called the **least upper bound** of K if
   - x is an upper bound for K, and
   - given any upper bound u for K, $x \leq u$.

   In this case, x is denoted by lub K.
3. The least upper bound for K is called the **greatest element** of K if $x \in K$.

4.2.20 EXERCISE

Using Definition 4.2.19 as your model, construct definitions for **lower bound, bounded below, greatest lower bound,** and **least element** of a subset K of a partially ordered set A.[5] □

4.2.21 EXERCISE

Consider $\mathbb{R}$ under the customary ordering $\leq$.

1. Let $K = [-3, 3]$. Find four upper bounds for K. Does K have a least upper bound? A greatest element?
2. Find an example of a subset K of $\mathbb{R}$ in which K has no lower bound.
3. Find an example of a subset K of $\mathbb{R}$ in which K has no least element but has a greatest lower bound. □

4.2.22 THEOREM

Let A be a partially ordered set. Let K be a nonempty subset of A. If K has a least upper bound, it is unique. □

4.2.23 DEFINITION

A partially ordered set in which every nonempty subset that is bounded above has a least upper bound is said to have the **least upper bound property**.

4.2.24 EXAMPLE

1. You will show in Problem 11 at the end of the chapter that for any set X, $\mathcal{P}(X)$ has the least upper bound property.
2. The most important example of an ordered set with the least upper bound property is $(\mathbb{R}, \leq)$. You should think about this for a while to see if you believe that it is true. It is an axiom of the real number system that $\mathbb{R}$, ordered as usual, has the least upper bound property. (See Chapter 8 for more details.) ■

4.2.25 LEMMA

Let A be a partially ordered set and let K be a subset of A. Define

$$\mathcal{L}_K = \{x \in A : x \text{ is a lower bound for } K\}.$$

[5] The least upper bound and greatest lower bound of K are also commonly called the **supremum** and the **infimum** of K. These are denoted by sup K and inf K, respectively.

Suppose that $\mathcal{L}_K$ has a least upper bound. Show that the least upper bound x of $\mathcal{L}_K$ is the greatest lower bound of K. □

4.2.26 THEOREM

Let A be a partially ordered set that has the least upper bound property. Then every nonempty subset of A that is bounded below has a greatest lower bound. (Or we might say: Every partially ordered set with the least upper bound property also has the **greatest lower bound property**.)

(*Hint:* Use Lemma 4.2.25.) □

"Lemma" is another word for theorem, but it has the additional meaning that it is a theorem proved mostly as an aid to proving another, bigger theorem. The lemma may or may not be interesting in its own right.

4.3 Equivalence Relations

Useful relations on a set are often closely related to the way that we organize the set in our minds. For example, the relation "$\leq$" describes the ordering that we place on the real numbers. When we think of sets in everyday life, we may tend to divide them into categories. We divide the set of adult people into two categories: men and women. We divide the set of foods into the four basic food groups: cereals, meats, fruits and vegetables, and dairy products. We divide the set of college students into seniors, juniors, sophomores, and freshmen. We divide the set of integers into even and odd integers.

Such divisions are as useful in mathematics as they are in other endeavors; we will now consider them in mathematical terms. Notice that in each set above, all the categories are mutually exclusive (no member belongs to more than one category) and exhaustive (every member belongs to some category). The mathematical term for this is "partition." A partition of a set S is a collection of mutually exclusive and exhaustive subsets of S. Here are some formal definitions.

4.3.1 DEFINITION

Let S be a set. Let Ω be a collection of subsets of S. The elements of Ω are said to be **pairwise disjoint** if for all elements $A, B \in \Omega$, either $A = B$ or $A \cap B = \emptyset$.

You may wonder why we do not say "Let A and B be *distinct* elements of K. Then $A \cap B = \emptyset$." This is clearly equivalent and seems less confusing. Our phrasing is chosen to suggest a certain way of thinking about the idea. In practice, when we set out to show that a collection of sets is pairwise disjoint, we usually assume $A \cap B \neq \emptyset$ and then prove that A and B must

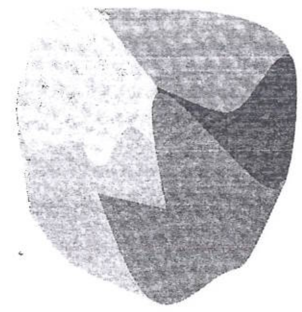


Figure 4.9 A partition of a set S

4.3.2 EXERCISE

Find an infinite collection of pairwise disjoint subsets of $\mathbb{R}$. □

4.3.3 DEFINITION

A collection Ω of nonempty subsets of a set S is said to be a **partition of S** provided that the elements of Ω are pairwise disjoint and their union is all of S. That is,

i. given A and $B \in \Omega$, either $A = B$ or $A \cap B = \emptyset$, and

ii. $\bigcup_{A \in \Omega} A = S$.

4.3.4 EXERCISE

I said above that a partition of S is a collection of mutually exclusive and exhaustive categories (subsets) of S. Which of the two provisions in the definition corresponds to mutual exclusivity? Which corresponds to the fact that the categories are exhaustive? □

4.3.5 EXERCISE (Partitions)

Construct the following examples of partitions.

1. Give an example of a partition of the set $S = \{1, 2, 3, 4\}$.
2. Give an example of a partition of $\mathbb{N}$ that has four elements.
3. Give an example of a partition of $\mathbb{R}^2$ that has infinitely many elements. □

Thus far we have been thinking about dividing a set into disjoint subsets. This division into categories can be seen in another light—in terms of relations. We can think of all members of a given category as being related to each other. This gives a relation; we take all possible ordered pairs of elements that come from the same category.

As you might suspect, the set-theoretic and relational interpretations of categorization are closely related. In fact, every collection of subsets of A (not just partitions) can be associated in a natural way with a relation on A. Conversely, every relation on A

4.3.6 DEFINITION

Let A be a set and Ω any subset of $\mathcal{P}(A)$. If a_1 and a_2 are elements of A, we will say that a_1 is related to a_2

if there exists an element $R \in \Omega$ that contains both a_1 and a_2.

This relation, $\sim_\Omega$, is called the **relation on A associated with Ω.**

4.3.7 EXERCISE

Let

$$A = \{1, 2, 3, 4, 5, 6\} \quad \text{and}$$
$$\Omega = \{\{1, 3, 4\}, \{2, 4\}, \{3, 4\}\}.$$

List the elements of $\sim_\Omega$. □

4.3.8 THEOREM

Let A be a set and let Ω be a subset of $\mathcal{P}(A)$. Then the relation $\sim_\Omega$ associated with Ω is symmetric. □

4.3.9 DEFINITION

Let A be a set. Let $\sim$ be any relation on A. Every $a \in A$ gives us a subset of A:

$$T_a = \{x \in A : a \sim x\}.$$

The set T_a is called the set of **relatives of a under $\sim$**. All of these subsets make up a collection $\Omega_\sim$ of subsets of A:

$$\Omega_\sim = \{T_a : a \in A\}.$$

$\Omega_\sim$ is called the **collection of subsets of A associated with $\sim$.**

4.3.10 EXERCISE

Let $A = \{1, 2, 3, 4, 5, 6\}$. Suppose

$$\sim = \{(1, 1), (2, 2), (2, 3), (2, 5), (3, 5), (4, 2), (4, 3), (4, 5), (5, 2), (5, 3), (5, 5)\}.$$

1. For each $a \in A$, find T_a.
2. Now find $\Omega_\sim$. □

4.3.11 PROBLEM

Let $A = \{1, 2, 3, 4, 5, 6\}$.

1. Consider the following subset of $\mathcal{P}(A)$:

$$\Omega = \{\{1, 2, 3, 4\}, \{5, 6\}\}.$$

Find

2. Consider the following relation on A:

$$\sim = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 2), (1, 4), (2, 1), (2, 4), (4, 1), (4, 2), (3, 6), (6, 3)\}.$$

Find $\Omega_\sim$. □

4.3.12 PROBLEM

Paul and Bettie have four sons: P. W. (age 12), Pat (age 10), Will (age 7), and Ben (age 4). In this family, person A is related to person B if A is older than B. (Bettie is older than Paul.)

What is the collection of subsets associated with the relation? □

4.3.13 PROBLEM

Poll between 5 and 10 people. From this set of people, form the following subsets.

- The set of all people who own cats.
- The set of all people who own gerbils.
- The set of people who do not have pets.

Describe the relation associated with this collection of subsets. Theorem 4.3.8 says that it should be symmetric. Is it reflexive? Antisymmetric? Transitive? Explain. □

4.3.14 PROBLEM

Give an example of a set A and a subset Ω of $\mathcal{P}(A)$ such that

1. the relation $\sim$ associated with Ω is reflexive.
2. the relation $\sim$ associated with Ω is not reflexive. □

Complete the statement of the following theorem, then prove it.

4.3.15 THEOREM

Let A be a set and let Ω be a subset of $\mathcal{P}(A)$.

If ____________, then the relation $\sim_\Omega$ is reflexive. □

4.3.16 THEOREM

Let A be a set and let Ω be a subset of $\mathcal{P}(A)$. Suppose that the elements of Ω are pairwise disjoint. Then the relation $\sim_\Omega$ associated with Ω is transitive. □

4.3.17 COROLLARY

Let S be a set. Let Ω be a partition of S. Then the relation on S associated with Ω is reflexive, symmetric, and transitive. □

"Corollary" is another word for theorem, but it has the additional meaning that its proof follows immediately from previously proved theorems.

4.3.18 DEFINITION

A relation on S that is reflexive, symmetric, and transitive is called an **equivalence relation**.

4.3.19 EXERCISE (Equivalence relations)

Give an example of an equivalence relation. (Be sure to specify the set on which the relation is defined.) □

4.3.20 LEMMA

Suppose A is a set. Let $\sim$ be an equivalence relation on A and let $a, b \in A$. Then

$$T_a = T_b \text{ if and only if } a \sim b.$$ □

We have seen that a partition of S yields an equivalence relation. Conversely, we have the following.

4.3.21 THEOREM

Let $\sim$ be an equivalence relation on a set S. Then $\Omega_\sim$ forms a partition of S. That is,

- $\bigcup_{x \in S} T_x = S$, and
- for x and y in S, either $T_x = T_y$, or $T_x \cap T_y = \emptyset$.

(*Hint:* This is a set-theoretic theorem. You are showing that sets are equal, so you will need to use element arguments just as you did in the chapter on set theory. For additional guidance on the proof of the second part, see the box on page 78.) □

By now it should be pretty clear that the study of equivalence relations on S is intimately related to the partitioning of S into disjoint subsets. In fact, *they are two faces of the same problem.* When we study partitions, we can view them mathematically in either way: We can bring the tools of set theory to bear when we think of a partition as a collection of subsets, or we can use what we know about relations to make sense of them! These two views complement each other, and it is useful to be able to cross over easily from one to the other.

To make this juggling act advance more smoothly, we introduce some more lan-

4.3.22 DEFINITION

Let S be a set, let $\sim$ be an equivalence relation on S, and let $x \in S$. Then T_x is called the **equivalence class of x under $\sim$.**

Following this, $\Omega_\sim$ is called the set of equivalence classes of S given by $\sim$ (or simply the **equivalence classes of** $\sim$.)

Lemma 4.3.20 and Theorem 4.3.21 explain why we introduced this new set of terms. Theorem 4.3.21 tells us that the T_x's are a set of mutually exclusive and exhaustive "categories" of S. Lemma 4.3.20 tells us what these "categories" (equivalence classes) are. Two elements of S fall into the same "category" (equivalence class) if and only if they are related to each other. All the elements in an equivalence class are related to one another. Any element outside a particular equivalence class is unrelated to elements in the equivalence class.

4.3.23 EXERCISE

Show that the following relations $\sim$ on the specified set S are equivalence relations. In each case do this in two ways:

- By identifying the equivalence classes and noting that they partition S.
- By showing directly that $\sim$ is reflexive, symmetric, and transitive.

1. $S = \{p : p \text{ is a person in Ohio}\}$. $A \sim B$ if A and B were born in the same year.
2. $S = \mathbb{Z}$. $a \sim b$ if $|a| = |b|$.
3. $S = \mathbb{Z}$. $a \sim b$ if $a - b$ is an integral multiple of 5. □

4.4 Graphs

Consider the digraph shown in Figure 4.3. Notice that it is possible to get from Dubuque to Chicago and back. However, it is only possible to go one way between Rochester and Philadelphia. If every possible route was two-way, it would make sense to eliminate the two arcs going different ways and replace them by a single undirected edge. The resulting diagram would become Figure 4.10.

If we think about what happened in mathematical terms, we started with a symmetric relation in which no element was related to itself. In such a relation, for distinct x and y, saying that x is related to y is the same as saying that y is related to x; therefore, it is sometimes convenient to simply conjoin the two ordered pairs (x, y) and (y, x) and think of them as a single *un*ordered pair $\{x, y\}$ (in effect, the two-element set containing x and y). So a symmetric relation in which no element is related to itself can be thought

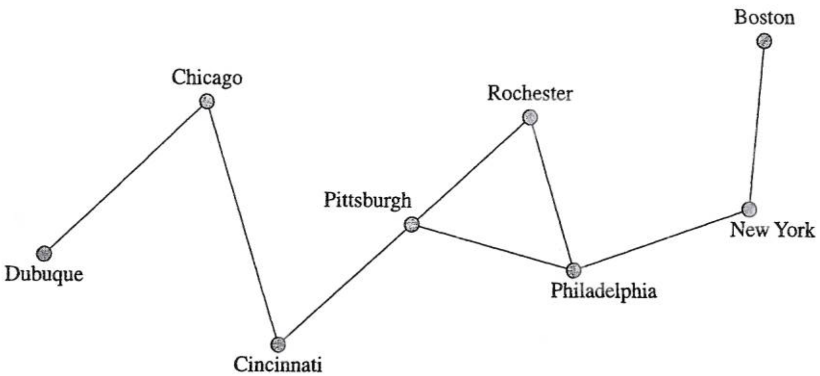


Figure 4.10 Two-way transportation network

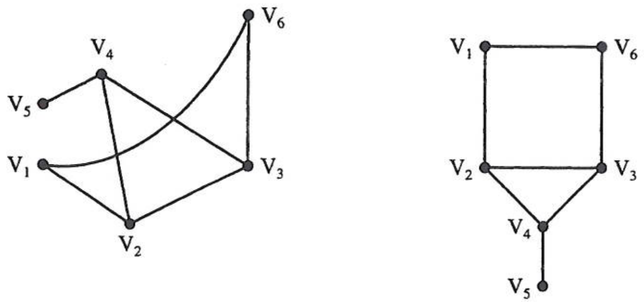


Figure 4.11 · A graph

4.4.1 DEFINITION

Let V be a set and E a set of unordered pairs of elements of V. Then the pair $G = (V, E)$ is called a **graph** on the set V.

The elements of the set V are called the **vertices of** G.

If v_1 and v_2 are vertices of G, and $\{v_1, v_2\} \in E$, we say that there is an **edge** in G joining v_1 and v_2. Thus the elements of the set E are called the **edges of** G.

Figure 4.11 shows a convenient way to represent graphs. The diagram shows the vertices and the edges between them. The placement of the vertices and the lengths of the edges do not matter. Crossings where there is not a vertex have no significance and

4.4.2 EXERCISE

For the graph shown in Figure 4.11, list the vertices and the edges. Verify that both diagrams give the same answer. □

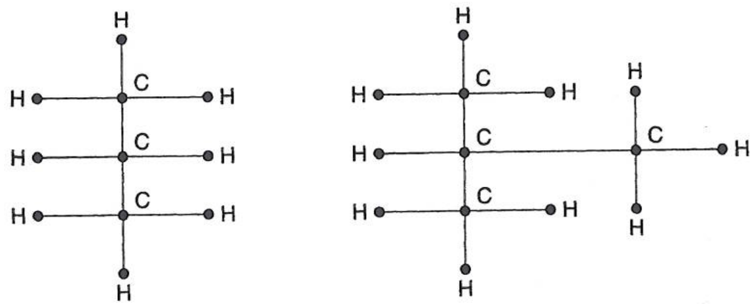
Graphs have many applications. Here are some examples that illustrate the variety of fields in which graph theory can be useful.

4.4.3 EXAMPLE

1. The two graphs shown in Figure 4.12 represent the chemical bonds of propane and isobutane molecules. (The vertices that are attached to four edges are carbon atoms, the vertices that are attached only to one edge are hydrogen atoms.)
2. Figure 4.13 shows the family tree of Indo-European language groups, as proposed by August Schleicher in 1862. English is part of the Germanic group. (Modern versions of the diagram are more complex and include additional groups unknown to Schleicher.)
3. Figure 4.14 shows an electrical resistor network. The vertices in the network are connected by imperfect conductors, called resistors. (Similar graphs can be drawn showing other kinds of electrical networks and computer networks.) ■

Remarks. Graph theory is a rich and varied subject. This will be a very brief treatment. To focus the discussion, I have decided to introduce only those ideas and theorems that I need to make it possible to discuss map coloring. In particular, I will limit the sorts of graphs that we look at.

1. Though the definition of graph makes perfect sense if the set of vertices is infinite, we will confine our discussion to talking about finite graphs. So whenever I say "Let $G = (V, E)$ be a graph" I really mean "Let V be a finite set and let $G = (V, E)$ be a graph on V."



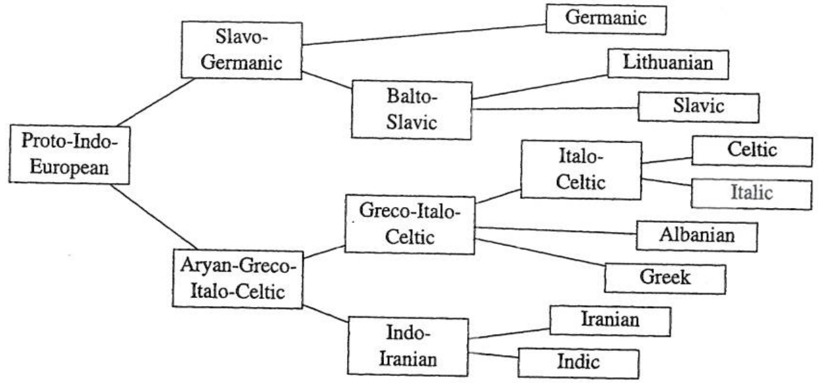


Figure 4.13 The descent of Indo-European languages

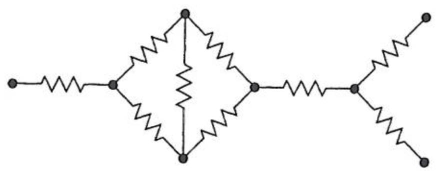


Figure 4.14 An electrical resistor network

2. Some discussions of graphs assume the possibility of an edge from a vertex to itself and of more than one edge joining the same pair of vertices. The definition I gave does not allow for either of these.

> **Something to keep in mind:** Because the graphs we will be dealing with have a finite number of vertices and, therefore, a finite number of edges, mathematical induction (on either the number of vertices or the number of edges) is very well suited to proving theorems about these graphs.

It would be an understatement to call graph theory a "definition-rich" subject. That is, a there are lots of words you need to know just to carry on a conversation about graph theory. Please bear with me while I define lots of terms. The good news is that most of the ideas are intuitive, and the definitions are easy to remember. (As you read, draw lots of pictures to help you understand the meanings of the terms.)

4.4.4 DEFINITION (Inside the graph)

Let $G = (V, E)$ be a graph.

1. Let u and v be vertices in G. If the pair $\{u, v\}$ is in E, then u and v are said to be **adjacent vertices**. The edge $\{u, v\}$ is said to be **incident** with the vertices u and v.
2. The **degree** of a vertex v is the number of edges incident with v. The degree of v is denoted by $\deg(v)$.
3. A vertex of degree one is called an **end vertex.**
4. If u, v, and w are vertices, and $\{u, v\}$, $\{v, w\}$, and $\{w, u\}$ are all edges, then u, v, and w are said to form a **triangle** in G.
5. Let $V^* \subseteq V$. If $E^* \subseteq E$, and the edges in E^* are incident only with vertices in V^*, then $G^* = (V^*, E^*)$ is said to be a **subgraph** of G.
6. Let $V^* \subseteq V$, and let E^* be the set of all edges that join pairs of vertices in V^*. Then $G^* = (V^*, E^*)$ is called the **subgraph of G generated by V^*.**

4.4.5 EXERCISE

For this exercise, refer once again to Figure 4.11. Answer the following questions about the graph G depicted there.

1. Find a pair of vertices of G that are adjacent and a pair of vertices that are not adjacent.
2. What edges of G are incident with v_3?
3. Which vertex in G has largest degree? Which vertex has smallest degree?
4. Does the G contain a triangle?
5. Draw the subgraph of G that is generated by the vertices $\{v_1, v_2, v_4, v_5\}$.
6. Draw the subgraph of G that is generated by the vertices $\{v_1, v_3, v_5, v_6\}$.
7. Find a subgraph of G that is *not* the subgraph generated by its set of vertices. □

4.4.6 THEOREM

Let $G = (V, E)$ be a graph. Let e be the number of edges of G. Then

$$\sum_{v \in V} \deg(v) = 2e.$$

□

4.4.7 DEFINITION (Moving around the graph)

Let $G = (V, E)$ be a graph.

1. Let u and v be vertices in G. A **walk** in G from u to v is an alternating list of vertices and edges in which each edge is incident with the vertices that come before and after it:

$$u, \{u, v_1\}, v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots, v_{n-1}, \{v_{n-1}, v\}, v.$$