2.2 Subsets

Often we deal with several sets at once and study the relationships between them. Here is one relationship that is probably already familiar to you.

2.2.1 DEFINITION

If A and S are sets, we say that S is a **subset** of A if every element of S is in A (which can be written in implication form as "If $x \in S$, then $x \in A$.") We denote this by $S \subseteq A$.

The next theorem shows that every nonempty set has at least two subsets.

2.2.2 THEOREM

For all sets X, $\emptyset \subseteq X$ and $X \subseteq X$. (*Hint:* Look at the implication given in the definition of subset. Explicitly write out the implications that you are meant to prove. What can you conclude?) □

2.2.3 EXERCISE

How many subsets does $\emptyset$ have? □

Notice that the statement "If $x \in S$, then $x \in X$" is an implication just like those that we studied in the last chapter. Thus to show that $S \subseteq X$, we must take an x for which the hypothesis is true (that is, an x in S) and show that the conclusion is also true for that x. That is, we must show $x \in X$. We do this so often in mathematics that there is a name for the process. It is called an **element argument**. To get started, try your hand at the following (very simple) element argument.

2.2.4 EXERCISE

Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (Notice that your goal is to prove that $A \subseteq C$. Therefore, according to the process described above, you begin your argument with "Let $x \in A$." At the end of your argument, you should be able to say: "Then $x \in C$.") □

2.2.5 DEFINITION

If B is a subset of X and $B \neq X$, then we say that B is a **proper subset** of X.

2.2.6 EXERCISE

Give two proper subsets of the set $\{1, 3, 5, 7, 9, 11\}$. □

Because a set is completely characterized by its elements, it is reasonable to say that two sets are the same set if they have the same elements. That is, every element of A is in B and every element of B is in A.