6. $A = \mathbb{R}$. $x \sim y$ if $x - y$ is rational. (You may assume here that the negative of a rational number is rational and the sum of two rational numbers is rational.)
7. $A = \{\ell : \ell \text{ is a line in the Cartesian plane}\}$. $x \sim y$ if x and y are parallel lines or if x and y are the same line. □

4.2 Orderings

In the previous section we mentioned the order $\leq$ on $\mathbb{R}$ as a familiar example of a relation. Order relations are fundamental to many branches of mathematics.

4.2.1 DEFINITION

A relation $\leq$ on a set A is called a **partial ordering** if $\leq$ is reflexive, antisymmetric, and transitive. A set together with a partial ordering on it is called a **partially ordered set**.[2] When we need to talk about a partially ordered set, we may refer to the *pair* $(A, \leq)$ or we may equivalently talk about the set A *under* the relation $\leq$.

Note that we use the symbol $\leq$ generically to refer to any relation that is reflexive, antisymmetric, and transitive. The underlying set may not be a set of real numbers; the symbol does not, unless specifically indicated, refer to the traditional ordering of the real numbers.

In Section 4.1 we said that we usually use the notation $a \sim b$ to indicate that the pair (a, b) is in a given relation. As the definition indicates, when that relation is a partial ordering, we customarily use $a \leq b$. In this context $\geq$, $<$, and $>$ have their usual meanings; for instance, $a > b$ means that $b \leq a$ and $b \neq a$.

4.2.2 EXAMPLE

Verify that the following are examples of partially ordered sets.

1. $\mathbb{R}$ under the traditional ordering of the real numbers.
2. For any set A, $\mathcal{P}(A)$ under $\subseteq$.
3. Any set A under the ordering: $a \leq b$ iff $a = b$. (We will call this a **totally unordered set**. Explain why this is a sensible use of language.) ■

The second and third of these examples are different from the first in a very important way.

4.2.3 DEFINITION

A partially ordered set A with partial order $\leq$ is said to be **totally ordered** if given any two elements a and b in A, either $a \leq b$ or $b \leq a$. In this case, $\leq$ is called a **total ordering**.

[2] Partially ordered sets are sometimes called **posets** for short.