> In the event that there are one-to-one mappings of S onto S', we usually show that $\langle S, *\rangle$ is not isomorphic to $\langle S', *'\rangle$ (if this is the case) by showing that one has some structural property that the other does not possess.

3.11 Example The sets $\mathbb{Z}$ and $\mathbb{Z}^+$ both have cardinality $\aleph_0$, and there are lots of one-to-one functions mapping $\mathbb{Z}$ onto $\mathbb{Z}^+$. However, the binary structures $\langle \mathbb{Z}, \cdot\rangle$ and $\langle \mathbb{Z}^+, \cdot\rangle$, where $\cdot$ is the usual multiplication, are not isomorphic. In $\langle \mathbb{Z}, \cdot\rangle$ there are two elements x such that $x \cdot x = x$, namely, 0 and 1. However, in $\langle \mathbb{Z}^+, \cdot\rangle$, there is only the single element 1. ▲

We list a few examples of possible structural properties and nonstructural properties of a binary structure $\langle S, *\rangle$ to get you thinking along the right line.

Possible Structural Properties

1. The set has 4 elements.
2. The operation is commutative.
3. $x * x = x$ for all $x \in S$.
4. The equation $a * x = b$ has a solution x in S for all $a, b \in S$.

Possible Nonstructural Properties

a. The number 4 is an element.
b. The operation is called "addition."
c. The elements of S are matrices.
d. S is a subset of $\mathbb{C}$.

We introduced the algebraic notions of commutativity and associativity in Section 2. One other structural notion that will be of interest to us is illustrated by Table 3.3, where for the binary operation $*''$ on the set $\{x, y, z\}$, we have $x *'' u = u *'' x = u$ for all choices possible choices, x, y, and z for u. Thus x plays the same role as 0 in $\langle \mathbb{R}, +\rangle$ where $0 + u = u + 0 = u$ for all $u \in \mathbb{R}$, and the same role as 1 in $\langle \mathbb{R}, \cdot\rangle$ where $1 \cdot u = u \cdot 1 = u$ for all $u \in \mathbb{R}$. Because Tables 3.1 and 3.2 give structures isomorphic to the one in Table 3.3, they must exhibit an element with a similar property. We see that $b * u = u * b = u$ for all elements u appearing in Table 3.1 and that \$ $*' u = u *'$ \$ $= u$ for all elements u in Table 3.2. We give a formal definition of this structural notion and prove a little theorem.

3.12 Definition Let $\langle S, *\rangle$ be a binary structure. An element e of S is an **identity element for** $*$ if $e * s = s * e = s$ for all $s \in S$. ■

3.13 Theorem **(Uniqueness of Identity Element)** A binary structure $\langle S, *\rangle$ has at most one identity element. That is, if there is an identity element, it is unique.

Proof Proceeding in the standard way to show uniqueness, suppose that both e and $\bar{e}$ are elements of S serving as identity elements. We let them compete with each other. Regarding e as an identity element, we must have $e * \bar{e} = \bar{e}$. However, regarding $\bar{e}$ as an identity element, we must have $e * \bar{e} = e$. We thus obtain $e = \bar{e}$, showing that an identity element must be unique. ◆