UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2400 — Real Analysis

Day of examination: Friday, August 17

Examination hours: 09 – 13

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems (1a, 1b, 1c, 2a, 2b, etc...) count for 10 points each. You have to explain all answers, and show enough details so that it is easy to follow your arguments. At the end of this document you will find some facts that might be handy. You may answer the exam in either English or Norwegian.

Problem 1

Let X be the space $X = C([0,1], \mathbb{R})$ of continuous real valued functions on the interval $[0,1]$. We equip X with the sup-norm, *i.e.*, for $f \in X$ we set

$$\|f\| = \sup_{x\in[0,1]} \{|f(x)|\},$$

and we get the induced metric $d(f,g) = \|f-g\|$ for all $f, g \in X$. Let x_0 and x_1 be two arbitrary points in $[0,1]$, and let $L : X \to \mathbb{R}$ be the map defined by

$$L(f) = f(x_0)\cdot f(x_1).$$

(a) Show that L has directional derivatives at each point $f \in X$.

(b) Show that L is differentiable at each point $f \in X$.

Solution:

(a) We have that

$$\begin{aligned}\lim_{t\to 0}\frac{L(f+tr)-L(f)}{t} &= \lim_{t\to 0}\frac{(f(x_0)+tr(x_0))(f(x_1)+tr(x_1))-f(x_0)f(x_1)}{t}\\ &= \lim_{t\to 0}\frac{tf(x_0)r(x_1)+tf(x_1)r(x_0)+t^2r(x_0)r(x_1)}{t}\\ &= f(x_0)r(x_1)+f(x_1)r(x_0).\end{aligned}$$

(b) By (a) we guess that the derivative is the map defined by

$$A(r) := f(x_0)r(x_1) + f(x_1)r(x_0).$$

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Since $f(x_0)$ and $f(x_1)$ are constants, A is clearly linear in r, and we have that

$$|A(r)| \leq \|f\| \cdot \|r\| + \|f\| \cdot \|r\| = 2\|f\| \cdot \|r\|,$$

so the map is also bounded.

It remains to check the decay of

$$\sigma(r) = L(f + r) - L(f) - A(r).$$

We have that

$$\begin{aligned}\sigma(r) &= (f(x_0) + r(x_0))(f(x_1) + r(x_1)) - f(x_0)f(x_1)\\ &\quad - f(x_0)r(x_1) - f(x_1)r(x_0)\\ &= r(x_0)r(x_1),\end{aligned}$$

and so $|\sigma(r)| \leq \|r\|^2$. This shows that $\frac{|\sigma(r)|}{\|r\|} \to 0$ as $\|r\| \to 0$.

Problem 2

Recall that a function $f : [-\pi, \pi] \to \mathbb{R}$ is *odd* if $f(-x) = -f(x)$ for all $x \in [-\pi, \pi]$.

(a) Let $f : [-\pi, \pi] \to \mathbb{R}$ be an odd continuous function. Show that the Fourier series of f is on the form

$$\sum_{n=1}^{\infty} b_n \sin(nx).$$

(b) Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by $f(x) = x$. Compute the Fourier series of f.

(c) Does the Fourier series of f converge uniformly on $[-\pi, \pi]$?

Solution:

(a) Since $\cos(nx)$ is an even function and $f(x)$ is an odd function, we have that $g_n(x) = f(x)\cos(nx)$ is an odd function, and we know that

$$\int_{-R}^{R} g_n(x)dx = 0$$

for any $R \geq 0$ as long as g_n is odd.

(b) Since f is an odd function, it suffices by (a) to compute the Fourier coefficients b_n. We use integration by parts, and we set $u(x) = x$ and $v' = \sin(nx)$, $u'(x) = 1$, $v(x) = \frac{-1}{n}\cos(nx)$. We get that

$$\begin{aligned}b_n &= \frac{1}{\pi}([\frac{-x}{n}\cos(nx)]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\sin(nx)dx)\\ &= \frac{1}{n\pi}(-\pi\cos(n\pi) - \pi\cos(-n\pi))\\ &= (-1)^{n+1}\frac{2}{n}.\end{aligned}$$

So the Fourier series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx).$$

(Continued on page 3.)

(c) We know that the Fourier series converges pointwise to the periodic function $\tilde{f}(x)$ defined as being equal to f on $(-\pi, \pi)$ but $\tilde{f}(-\pi) = \tilde{f}(\pi) = 0$. This function is not continuous, and so the convergence cannot be uniform, since the uniform limit of a sequence of continuous functions is continuous.

Problem 3

Let Y be a non-empty set, and let d_n be metrics on Y for $n = 1, 2, 3, ...$ Suppose that there exists a constant $C \geq 1$ such that $d_m(x, y) \leq C \cdot d_k(x, y)$ for all $k, m = 1, 2, 3, ...,$ and all $x, y \in Y$. We let $d : Y \times Y \to \mathbb{R}$ be the function

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \cdot d_n(x, y),$$

for all $x, y \in Y$.

(a) Show that $d(x, y) < \infty$ for all $x, y \in Y$.

(b) Show that d is a metric.

(c) Show that if $E = \{y_j\}_{j \in \mathbb{N}} \subset Y$ is a Cauchy sequence with respect to one of the metrics d_k, then E is a Cauchy sequence with respect to d.

Solution:

(a) We have that

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \cdot d_n(x, y) \leq \sum_{n=1}^{\infty} 2^{-n} \cdot C \cdot d_1(x, y) = C \cdot d_1(x, y).$$

(b) This is straight forward (don't write that on an exam).

(c) Given $\epsilon > 0$ there exists N such that $d_k(x_l, x_m) \leq \epsilon / C$ for all $l, m \geq N$. Then for $l, m \geq N$ we get that

$$d(x_l, x_m) \leq \sum_{n=1}^{\infty} 2^{-n} \cdot C \cdot d_k(x_l, x_m) \leq C \cdot d_k(x_l, x_m) \sum_{n=1}^{\infty} 2^{-n} \leq \epsilon.$$

Problem 4

Let H be a complete inner product space over $\mathbb{R}$, with an inner product $\langle \cdot, \cdot \rangle$. Suppose that $l : H \to \mathbb{R}$ is a continuous linear map which is not identically zero. We set

$$\mathrm{Ker}(l) := \{\mathbf{u} \in H : l(\mathbf{u}) = 0\},$$

and further we let W denote the orthogonal complement

$$W = \{\mathbf{u} \in H : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{v} \in \mathrm{Ker}(l)\}.$$

(a) Prove that W does not consist only of the zero vector. (It is a fact, which you can take for granted, that any vector $\mathbf{u} \in H$ may be written uniquely as a sum $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathrm{Ker}(l)$ and $\mathbf{w} \in W$.)

(Continued on page 4.)

(b) Prove that the restriction $l : W \to \mathbb{R}$ is a bijective continuous map, *i.e.*, there exists a continuous linear map $g : \mathbb{R} \to W$ such that $g(l(\mathbf{u})) = \mathbf{u}$ for all $\mathbf{u} \in W$. Prove that there exists a vector $\mathbf{w} \in W$ such that $l(\mathbf{u}) = \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u} \in H$.

Solution:

(a) Suppose W consists only of the zero vector. Then any $\mathbf{u}$ can be written $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathrm{Ker}(l)$ and $\mathbf{w} = 0$, and so $l(\mathbf{u}) = l(\mathbf{v}) + l(\mathbf{0}) = 0$. This contradicts the assumption that l is not identically zero.

(b) We show first that W is 1-dimensional. If not, there exist two non co-linear vectors $\mathbf{w_1}, \mathbf{w_2} \in W$, and then real numbers α_1, α_2 such that $\alpha_1 l(\mathbf{w_1}) + \alpha_2 l(\mathbf{w_2}) = 0$. By linearity we get that $\mathbf{w_3} = \alpha_1 \mathbf{w_1} + \alpha_2 \mathbf{w_2} \in \mathrm{Ker}(l)$. By assumption, $\langle \mathbf{w_3}, \mathbf{w_3} \rangle \neq 0$, which is a contradiction.

So W is spanned by a single non-zero vector $\mathbf{w}'$, and we may define $g(x) = x/(l(\mathbf{w}')) \cdot \mathbf{w}'$.

Finally we set $\mathbf{w} = \frac{l(\mathbf{w}')\mathbf{w}'}{\|\mathbf{w}'\|^2}$. Any vector in W may be written as $t \cdot \mathbf{w}'$ for $t \in \mathbb{R}$, and we see that

$$\langle t\mathbf{w}', \mathbf{w} \rangle = \frac{tl(\mathbf{w}')}{\|\mathbf{w}'\|^2} \langle \mathbf{w}', \mathbf{w}' \rangle = tl(\mathbf{w}') = l(t\mathbf{w}').$$

THE END

Some facts:

Recall that if $f : [-\pi, \pi] \to \mathbb{R}$ is a continuous function, then the (real) Fourier series of f is given by

$$\frac{a_0}{2} + \sum_n a_n \cos(nx) + \sum_n b_n \sin(nx),$$

where a_n is given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

for $(n = 0, 1, 2, ...)$, and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx,$$

for $n = 1, 2, 3, ...$

Recall also that if $L : X \to Y$ is a map between linear spaces, then the directional derivative $L'(f; r)$ at a point $f \in X$ in the direction $r \in X$ is given by

$$L'(f; r) = \lim_{t \to 0} \frac{L(f + tr) - L(f)}{t},$$

provided that the limit exists.