MAT2400: Mandatory Assignment I, Spring 2013

Deadline: *You must turn in your paper before 2.30 p.m., Thursday, March 7, 2013, in the designated area on the 7th floor of NHA. Remember to use the official front page available on the 7th floor and at*

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/obligforside.pdf

If you due to illness or other circumstances want to extend the deadline, you must apply for an extension to studieinfo@math.uio.no *Remember that illness has to be documented by a medical doctor! See*

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/index.html

for more information about the rules for mandatory assignments.

Instructions: *The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the assignment accepted, you need a score of at least 60%. In the evaluation, credit will be given for a clear and well-organized presentation. All questions (points 1a), 1b) etc.) have equal weight. Students who do not get their original paper accepted, but who have made serious and documented attempts to solve the problems, will get* one *chance of turning in an improved version.*

In solving the problems you may collaborate with others and use tools of all kinds. However, the paper you turn in should be written by you *(by hand or computer) and should reflect* your *understanding of the material. If we are not convinced that you understand your own paper, we may ask you to give an oral presentation.*

Problem 1. Assume that X is a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a *semi-metric* if

(i) For all $x, y \in X$, $d(x, y) \geq 0$ with equality when $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

(the only difference from a metric is that $d(x, y)$ may be 0 even when $x \neq y$). The pair (X, d) is then called a *semi-metric space*.

a) Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1|$. Show that d is a semi-metric, but not a metric on $\mathbb{R}^2$.

A sequence $\{x_n\}$ in a semi-metric space X is said to *converge* to a point $x \in X$ if for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$ for all $n \geq N$.

b) In a semi-metric space a sequence may converge to more than one point. If X is as in part a), find all the points the sequence $\{(\frac{1}{n}, 0)\}$ converges to.

c) Assume that (X, d) is a semi-metric space, and write $x \sim y$ if $d(x, y) = 0$. Show that $\sim$ is an equivalence relation on X (recall section 1.5 in the notes). Describe the equivalence classes of the space in part a).

d) Show that if $x_1 \sim x_2$ and $y_1 \sim y_2$, then $d(x_1, y_1) = d(x_2, y_2)$.

e) Let $[x]$ denote the equivalence class of an element $x \in X$, and let $\hat{X}$ be the set of all equivalence classes. Explain that we can define a function $\hat{d} : \hat{X} \times \hat{X} \to \mathbb{R}$ by

$$\hat{d}([x], [y]) = d(x, y)$$

and show that $(\hat{X}, \hat{d})$ is a metric space.

Problem 2. In this problem, a function $\phi : [0, \infty) \to [0, \infty)$ is called a *modifying function* if

(i) $\phi(0) = 0$

(ii) ϕ is strictly increasing

(iii) ϕ is subadditive; i.e. $\phi(s + t) \leq \phi(s) + \phi(t)$ for all $s, t \in [0, \infty)$.

a) Show that the function $\phi : [0, \infty) \to [0, \infty)$ given by

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \\ t + 1 & \text{if } t > 0 \end{cases}$$

is a modifying function.

b) Show that if a modyfying function is continuous at 0, then it is uniformly continuous.

c) Show that if d is a metric on a set X, and ϕ is a modifying function, then the function $d_\phi : X \times X \to \mathbb{R}$ defined by

$$d_\phi(x, y) = \phi(d(x, y))$$

is also a metric on X.

d) Show that if ϕ is a continuous modifying function, then (X, d) and (X, d_ϕ) have the same open, closed and compact sets.

e) Show that if $f : [0, \infty) \to [0, \infty)$ is a differentiable function such that

(i) $f(0) = 0$.

(ii) f' is continuous, strictly positive and decreasing

then f is a modifying function.

f) Show that if d is a metric on X, then $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is also a metric on X, and that d and $\tilde{d}$ have the same open sets. (Note that the metric $\tilde{d}$ is bounded since $\tilde{d}(x, y) < 1$ for all $x, y \in X$, and hence it always possible to replace a metric by a bounded metric with many of the same properties.)