MAT2400: Mandatory Assignment II, Spring 2013

<u>Deadline</u>: *You must turn in your paper before 2.30 p.m., Thursday, May 2, 2013, in the designated area on the 7th floor of NHA. Remember to use the official front page available on the 7th floor and at*

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/obligforside.pdf

If you due to illness or other circumstances want to extend the deadline, you must apply for an extension to studieinfo@math.uio.no. *Remember that illness has to be documented by a medical doctor! See*

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/index.html

for more information about the rules for mandatory assignments.

<u>Instructions:</u> *The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the assignment accepted, you need a score of at least 60%. In the evaluation, credit will be given for a clear and well-organized presentation. All questions (points 1a), 1b) etc.) have equal weight. Students who do not get their original paper accepted, but who have made serious and documented attempts to solve the problems, will get* one *chance of turning in an improved version. In solving the problems you may collaborate with others and use tools of all kinds. However, the paper you turn in should be written by* <u>you</u> *(by hand or computer) and should reflect* <u>your</u> *understanding of the material. If we are not convinced that you understand your own paper, we may ask you to give an oral presentation.*

Problem 1. In this problem, $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two measure spaces. We assume that X and Y are disjoint and let $Z = X \cup Y$.

a) Show that
$$\mathcal{C} = \{A \cup B \,:\, A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$
is a σ-algebra on Z.

b) Define $\lambda : \mathcal{C} \to \bar{R}_+$ by $\lambda(A \cup B) = \mu(A) + \nu(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show that λ is a measure.

c) Assume that $f : X \to \bar{\mathbb{R}}$ and $g : Y \to \bar{\mathbb{R}}$ are two measurable functions (with respect to $\mathcal{A}$ and $\mathcal{B}$, respectively), and define $h \colon Z \to \bar{\mathbb{R}}$ by
$$h(z) = \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in Y \end{cases}$$
Show that h is measurable (with respect to $\mathcal{C}$).

Problem 2. In this problem, $(H, \langle \cdot, \cdot \rangle)$ is a real Hilbert space (i.e., H is a complete inner product space over $\mathbb{R}$), and $\| \cdot \|$ is the norm generated by $\langle \cdot, \cdot \rangle$.

a) Prove the *parallelogram law*

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

for all $\mathbf{u}, \mathbf{v} \in H$.

A nonempty subset K of H is called *convex* if for all $\mathbf{u}, \mathbf{v} \in K$ and all $\alpha \in [0, 1]$,

$$\alpha\mathbf{u} + (1 - \alpha)\mathbf{v} \in K$$

Intuitively this means that if $\mathbf{u}$ and $\mathbf{v}$ are elements in K, then the line segment connecting $\mathbf{u}$ and $\mathbf{v}$ also belongs to K.

b) Show that any subspace V of H is convex.

c) Show that any open ball $\mathrm{B}(\mathbf{a}, r)$ in H is convex.

From now on we assume that K is a nonempty, closed, convex subset of H and that $\mathbf{a} \in H$. Let

$$\beta = \inf\{\|\mathbf{x} - \mathbf{a}\| \, : \, \mathbf{x} \in K\}$$

d) Explain that it is possible to find a sequence $\{\mathbf{x}_n\}$ of points in K such that $\|\mathbf{x}_n - \mathbf{a}\| \to \beta$.

e) Show that $\{\mathbf{x}_n\}$ is a Cauchy sequence. (*Hint:* Show that by the parallelogram law

$$\|\mathbf{x}_n - \mathbf{x}_m\|^2 = 2\|\mathbf{x}_n - \mathbf{a}\|^2 + 2\|\mathbf{x}_m - \mathbf{a}\|^2 - 4\|\frac{1}{2}\mathbf{x}_n + \frac{1}{2}\mathbf{x}_m - \mathbf{a}\|^2 .)$$

f) Explain that $\{\mathbf{x}_n\}$ converges to a point $\mathbf{x} \in K$ such that $\|\mathbf{x} - \mathbf{a}\| = \beta$.

g) Show that $\mathbf{x}$ is the *only* point in K whose distance to K is β. (*Hint:* Assume that $\mathbf{y}$ is another point in K such that $\|\mathbf{y} - \mathbf{a}\| = \beta$ and show that by the parallelogram law

$$\|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x} - \mathbf{a}\|^2 + 2\|\mathbf{y} - \mathbf{a}\|^2 - 4\|\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} - \mathbf{a}\|^2 .)$$

h) Assume now that K is a closed *subspace* of H. Show that $\mathbf{a} - \mathbf{x}$ is orthogonal to all $\mathbf{u} \in K$.

Good Luck!