Solution to exam in MAT2400, Spring 2013

Problem 1: To prove pointwise convergence, we show that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = 0$ for all $x \in [0,1]$. For $x = 0$, this is obvious as $f_n(0) = 0$ for all n, and for $x \neq 0$, we use L'Hôpitals rule (remember to differentiate with respect to n and not x):

$$\lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} \overset{L'H}{=} \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = 0$$

To check uniform convergence, we find the maximal distance between f_n and 0 for each n. Differentiating f_n, we get

$$f_n'(x) = ne^{-nx^2} + nxe^{-nx^2}(-2nx) = ne^{-nx^2}(1-2nx^2)$$

which shows that the maximal distance between f_n and 0 is achieved at $x = \frac{1}{\sqrt{2n}} \in [0,1]$. Since

$$f_n(\frac{1}{\sqrt{2n}}) = n\frac{1}{\sqrt{2n}}e^{-n\cdot\frac{1}{2n}} = \sqrt{\frac{n}{2}}e^{-\frac{1}{2}} \to \infty$$

the convergence is not uniform.

Problem 2: We must show that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that when $n \geq N$, $|(f+g)(x) - (f_n + g_n)(x)| < \epsilon$ for all $x \in X$. Since the original sequences $\{f_n\}$ and $\{g_n\}$ converges uniformly to f and g, respectively, there are numbers $N_1, N_2 \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \frac{\epsilon}{2}$ for all $x \in X$ whenever $n \geq N_1$ and $|g(x) - g_n(x)| < \frac{\epsilon}{2}$ for all $x \in X$ whenever $n \geq N_2$. If we define $N = \max\{N_1, N_2\}$, the triangle inequality tells us that

$$|(f+g)(x) - (f_n+g_n)(x)| \leq |f(x) - f_n(x)| + |g(x) - g_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \in X$ and all $n \geq N$. Hence $\{f_n + g_n\}$ converges uniformly to $f + g$.

Problem 3: a) Assume first that f is nonnegative. Then there is an increasing sequence $\{g_n\}$ of nonnegative, simple functions converging pointwise to f. Since $\int f\,d\mu = \lim_{n\to\infty} \int g_n\,d\mu$ and $f \geq g_n$, we get $\|f - g_n\|_1 = \int |f - g_n|\,d\mu = \int (f - g_n)\,d\mu = \int f\,d\mu - \int g_n\,d\mu \to 0$ (one may also use Lebesgue's Dominated Convergence Theorem with f as dominating function).

For a general integrable function f, we split it as a difference $f = f_+ - f_-$ between two nonnegative, integrable functions, and approximate these by sequences $\{g_n^+\}$ and $\{g_n^-\}$ as above. Then

$$\|f - (g_n^+ - g_n^-)\|_1 \leq \|f_+ - g_n^+\|_1 + \|f_- - g_n^-\|_1 \to 0$$

Since $g_n^+ - g_n^-$ is a simple function (it takes only finitely many values), this proves the statement.

b) Let $B = \{x \in \mathbb{R} \mid h(x) \neq \mathbf{1}_A(x)\}$. Then

$$\|h(x) - \mathbf{1}_A(x)\|_1 = \int |h - \mathbf{1}_A|\, d\mu \leq \int \mathbf{1}_B \, d\mu = \mu(B) < \epsilon$$

c) Assume that $\epsilon > 0$, and let $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ be a simple function in $L^1(\mu)$. Let M be a number larger than all $|a_i|$, $i = 1, 2, \ldots, n$, and for each i choose a continuous function $h_i : \mathbb{R} \to [0,1]$ such that $h_i = \mathbf{1}_{A_i}$ except on a set of measure less that $\frac{\epsilon}{Mn}$. Note that by b), $\|h_i - \mathbf{1}_{A_i}\|_1 < \frac{\epsilon}{Mn}$. The function $h = \sum_{i=1}^n a_i h_i$ is continuous and

$$\|h - g\|_1 = \|\sum_{i=1}^n a_i(h_i - \mathbf{1}_{A_i})\|_1 \leq \sum_{i=1}^n |a_i| \|h_i - \mathbf{1}_{A_i}\|_1 < n \cdot M \cdot \frac{\epsilon}{Mn} = \epsilon$$

d) If $f \in L^1(\mu)$, we know that there exists a sequence $\{g_n\}$ of simple functions converging to f in $L^1(\mu)$-norm. For each n, we know that there is continuous function h_n such that $\|g_n - h_n\|_1 < \frac{1}{n}$. But then

$$\|f - h_n\|_1 \leq \|f - g_n\|_1 + \|g_n - h_n\|_1 \to 0$$

Problem 4: a) We have

$$0 \leq \lim_{n\to\infty} |\langle \mathbf{u}_n, \mathbf{v}_n\rangle - \langle \mathbf{u}, \mathbf{v}\rangle| = \lim_{n\to\infty} |\langle \mathbf{u}_n, \mathbf{v}_n\rangle - \langle \mathbf{u}, \mathbf{v}_n\rangle + \langle \mathbf{u}, \mathbf{v}_n\rangle - \langle \mathbf{u}, \mathbf{v}\rangle| \leq$$

$$\begin{aligned}
\lim_{n\to\infty} |\langle \mathbf{u}_n, \mathbf{v}_n\rangle - \langle \mathbf{u}, \mathbf{v}_n\rangle| + \lim_{n\to\infty} |\langle \mathbf{u}, \mathbf{v}_n\rangle - \langle \mathbf{u}, \mathbf{v}\rangle| &= \\
= \lim_{n\to\infty} |\langle \mathbf{u}_n - \mathbf{u}, \mathbf{v}_n\rangle| + \lim_{n\to\infty} |\langle \mathbf{u}, \mathbf{v}_n - \mathbf{v}\rangle| &\leq \\
\leq \lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\|\|\mathbf{v}_n\| + \lim_{n\to\infty} \|\mathbf{u}\|\|\mathbf{v}_n - \mathbf{v}\| &= 0
\end{aligned}$$

since $\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\| = \lim_{n\to\infty} \|\mathbf{v}_n - \mathbf{v}\| = 0$ and $\lim_{n\to\infty} \|\mathbf{v}_n\| = \|\mathbf{v}\|$ (continuity of norm).

b) Let $\mathbf{u}_n = \sum_{i=1}^n \alpha_i \mathbf{e}_i$ and $\mathbf{v}_n = \sum_{j=1}^n \beta_j \mathbf{e}_j$ and note that

$$\langle \mathbf{u}_n, \mathbf{v}_n\rangle = \sum_{1\leq i,j\leq n} \alpha_i \beta_j \langle \mathbf{e}_i, \mathbf{e}_j\rangle = \sum_{i=1}^n \alpha_i \beta_i$$

where we have used the orthonormality. By part a), we get

$$\langle \mathbf{u}, \mathbf{v}\rangle = \lim_{n\to\infty} \langle \mathbf{u}_n, \mathbf{v}_n\rangle = \lim_{n\to\infty} \sum_{i=1}^n \alpha_i \beta_i = \sum_{i=1}^\infty \alpha_i \beta_i$$

c) We prove (I) by induction on n. For $n = 1$, we have $A(\alpha_1 \mathbf{u}_1) = \alpha_1 A(\mathbf{u}_i)$ which is just condition (i). Assume that the assertion is proved for some number n, then

$$A(\sum_{i=1}^{n+1} \alpha_i \mathbf{u}_i) = A(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i + \alpha_{n+1}\mathbf{u}_{n+1}) \overset{(ii)}{=} A(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i) + A(\alpha_{n+1}\mathbf{u}_{n+1}) =$$

$$\overset{\text{ind. hyp.+(i)}}{=} \sum_{i=1}^{n} \alpha_i A(\mathbf{u}_i) + \alpha_{n+1} A(\mathbf{u}_{n+1}) = \sum_{i=1}^{n+1} \alpha_i A(\mathbf{u}_i)$$

shows that it holds for $n + 1$.

To prove (II), just note that

$$A(\mathbf{u} - \mathbf{v}) = A(\mathbf{u} + (-1)\mathbf{v}) = A(\mathbf{u}) + A((-1)\mathbf{v}) =$$

$$= A(\mathbf{u}) + (-1)A(\mathbf{v}) = A(\mathbf{u}) - A(\mathbf{v})$$

d) We have $|A(\mathbf{u}) - A(\mathbf{v})| = |A(\mathbf{u} - \mathbf{v})| \leq M\|\mathbf{u} - \mathbf{v}\|$. Given $\epsilon > 0$, put $\delta = \frac{\epsilon}{M}$. Then $|A(\mathbf{u}) - A(\mathbf{v})| < \epsilon$ whenever $\|\mathbf{u} - \mathbf{v}\| < \delta$, and hence A is uniformly continuous.

e) We have

$$A(\sum_{i=1}^{n} \beta_i \mathbf{e}_i) = \sum_{i=1}^{n} \beta_i A(\mathbf{e}_i) = \sum_{i=1}^{n} \beta_i^2$$

On the other hand

$$A(\sum_{i=1}^{n} \beta_i \mathbf{e}_i) \leq M\|\sum_{i=1}^{n} \beta_i \mathbf{e}_i\| = M\left(\sum_{i=1}^{n} \beta_i^2\right)^{\frac{1}{2}}$$

and thus $\sum_{i=1}^{n} \beta_i^2 \leq M\left(\sum_{i=1}^{n} \beta_i^2\right)^{\frac{1}{2}}$ which implies $\left(\sum_{i=1}^{n} \beta_i^2\right)^{\frac{1}{2}} \leq M$. Since this holds for all $n \in \mathbb{N}$, we must have $\left(\sum_{i=1}^{\infty} \beta_i^2\right)^{\frac{1}{2}} \leq M$.

f) Since H is complete, it suffices to show that the sequence of partial sums $\mathbf{s}_n = \sum_{i=1}^{n} \beta_i \mathbf{e}_i$ is a Cauchy sequence. If $m > n$, we have

$$\|\mathbf{s}_m - \mathbf{s}_n\|^2 = \|\sum_{i=n+1}^{m} \beta_i \mathbf{e}_i\|^2 = \langle \sum_{i=n+1}^{m} \beta_i \mathbf{e}_i, \sum_{i=n+1}^{m} \beta_i \mathbf{e}_i \rangle = \sum_{i=n+1}^{m} \beta_i^2 \leq \sum_{i=n+1}^{\infty} \beta_i^2$$

and since $\sum_{i=1}^{\infty} \beta_i^2$ converges, we can get the last term as small as we want by choosing n sufficiently large. Hence $\{\mathbf{s}_n\}$ is a Cauchy sequence, and the series $\sum_{i=1}^{\infty} \beta_i \mathbf{e}_i$ converges.

g) Since $\{\mathbf{e}_i\}_{i\in\mathbb{N}}$ is a basis, any element $\mathbf{x} \in H$ can be written as a linear combination $\mathbf{x} = \sum_{i=1}^{\infty} \alpha_i \mathbf{e}_i$. By b) above,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \sum_{i=1}^{\infty} \alpha \mathbf{e}_i, \sum_{j=1}^{\infty} \beta_j \mathbf{e}_j \rangle = \sum_{i=1}^{\infty} \alpha_i \beta_i$$

On the other hand, since A is continuous, we have

$$A(\mathbf{x}) = A(\sum_{i=1}^{\infty} \alpha_i \mathbf{e}_i) = \lim_{n\to\infty} A(\sum_{i=1}^{n} \alpha_i \mathbf{e}_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i A(\mathbf{e}_i) =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \beta_i = \sum_{i=1}^{\infty} \alpha_i \beta_i$$

Hence $A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x} \in H$.