Ark7: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1:

a) Recall Abel's formula for partial summation (**lemma 4.4.1** page 94 in Tom's notes), where $s_N = \sum_{n=0}^{N} a_n$:

$$\sum_{n=0}^{N} a_n b_n = s_N b_N + \sum_{n=0}^{N-1} s_n (b_n - b_{n+1}). \tag{$\star$}$$

Let now $M < N$ be another integer. We have

$$\sum_{n=0}^{M} a_n b_n = s_M b_M + \sum_{n=0}^{M-1} s_n (b_n - b_{n+1}), \tag{$\star\star$}$$

and subtraction equation ★★ from equation ★, we get

$$\sum_{n=M+1}^{N} a_n b_n = s_N b_N - s_M b_M + \sum_{n=M}^{N-1} s_n (b_n - b_{n+1}). \tag{$\star\star\star$}$$

b) From ★★★ we get by the triangle inequality and the facts that $b_n \geq b_{n+1} \geq 0$

$$\begin{aligned} \sum_{n=M+1}^{N} |a_n b_n| &\leq |s_N b_N| + |s_M b_M| + \sum_{n=M}^{N-1} |s_n| \,(b_n - b_{n+1}) \qquad (❀) \\ &\leq A b_N + A b_M + A \sum_{n=M}^{N-1} (b_n - b_{n+1}) \\ &= A b_N + A b_M + A(b_M - b_N) = 2 A b_M. \end{aligned}$$

where A is the bound for the partial sums of a_n, *i.e.*, $s_N = \left| \sum_{n=0}^{N} a_n \right| \leq A$ for all N. (Which was given the obviuosly bad name M in the problem). We have used the equality: $\sum_{n=M}^{N-1} (b_n - b_{n+1}) = b_M - b_N$ resulting from the telescoping propety; all the terms in the sum, except the two to the right in the formula, cancels.

c) Given $\epsilon > 0$. As b_n tends to zero when $n \to \infty$, we may find N_0 such that $b_n < \epsilon/2A$ for $n > N_0$, but then by the inequality ❀ we obtain

$$\sum_{n=M+1}^{N} |a_n b_n| \le 2Ab_M \le \epsilon$$

if $N, M > N_0$, and $\sum_{n=0}^{\infty} a_n b_n$ is Cauchy and converges.

d) Take $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum_{n=0}^{\infty} a_n$ converges by Liebnitz' criterion for alternating sums, but $a_n b_n = \frac{1}{n}$ and the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges. ❏

PROBLEM 2:

a) The formula

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

is well known, and follows from the addition formula for cosinus:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

and the facts that $\sin x$ is an odd function and $\cos x$ an even one.

b) This is just the previous formula with $\alpha = kx$ and $\beta = x/2$.

c) Use the formula in b) and the telescoping property to obtain the formula in the problem:

$$\sum_{k=1}^{n} 2 \sin kx \sin \frac{x}{2} = \cos \frac{x}{2} - \cos{(n + \frac{1}{2})x}.$$

Then, dividing by $2 \sin \frac{x}{2}$ — which we suppose is different from zero — and using the triangel inequality, we get

$$\left| \sum_{k=1}^{n} \sin kx \right| \le \frac{\left| \cos \frac{x}{2} - \cos{(n + \frac{1}{2})x} \right|}{2 \left| \sin \frac{x}{2} \right|} \le \frac{1}{\left| \sin \frac{x}{2} \right|}$$

d) Since $a > 0$ the sequence $\{k^{-a}\}$ decreases monotonically to zero. We saw that $\sum_{k=1}^{n} \sin kx$ is bounded. Hence Dirichlet's criterion gives us the convergence. ❏

PROBLEM 7:

a) We have to show that if $\{x_n\}$ and $\{y_n\}$ are two elements in l_1, then their sum $\{x_n + y_n\}$ is there also. That is, if the two former sequences are absolutely convergent, then the latter is. This is well known, and follows from the comparison test and the triangle inequality:

$$|x_n + y_n| \le |x_n| + |y_n| .$$

If a is a scalar, it is clear that $\{ax_n\}$ is in l_1 when $\{x_n\}$ is there.
b) Let a be a scalar. Then $\sum_{n=1}^{\infty}|ax_n| = |a|\sum_{n=1}^{\infty}|x_n|$. We have

$$\sum_{n=1}^{\infty}|x_n + y_n| \leq \sum_{n=1}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n|$$

by the triangle inequality. Finally if $\sum_{n=1}^{\infty}|x_n| = 0$, clearly $x_n = 0$ for all n.
c) That the e_i's form a basis, follows if we can show that $\{x_n\} = \sum_{k=1}^{\infty} x_k e_k$. The partial sum is *the sequence* $\sum_{k=1}^{N} x_k e_k$, which is equal to $\{x_1, x_2, \ldots, x_N, 0, 0, 0, \ldots\}$, *i.e.*, the sequence that is equal to x_n up to the index N and from then on equal to zero. The difference between $\{x_n\}$ and $\sum_{k=1}^{N} x_k e_k$ is the the sequence $z_N = \{0, 0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots\}$ whose norm is

$$\|z_N\| = \sum_{k=N+1}^{\infty} |x_k|$$

which tends to zero when N tends to ∞ since $\sum_{k=1}^{\infty}|x_k|$ converges since $\{x_n\}$ lies in l_1. ❑

PROBLEM 12 :

a) We have the Cauchy-Schwarz inequality from problem 11:

$$(\sum_{k=1}^{n} x_k y_k)^2 \leq (\sum_{k=1}^{n} x_k^2)(\sum_{k=1}^{n} y_k^2).$$

Since the two sequences $\sum_{k=1}^{\infty} x_k^2$ and $\sum_{k=1}^{\infty} y_k^2$ converge, we have

$$(\sum_{k=1}^{n} x_k y_k)^2 \leq (\sum_{k=1}^{n} x_k^2)(\sum_{k=1}^{n} y_k^2) \leq (\sum_{k=1}^{\infty} x_k^2)(\sum_{k=1}^{\infty}, y_k^2)$$

and letting n tend to ∞ we obtain:

$$(\sum_{k=1}^{\infty} x_k y_k)^2 \leq (\sum_{k=1}^{\infty} x_k^2)(\sum_{k=1}^{\infty} y_k^2).$$

b) The only problem in showing that l_2 is a vector space, is to see that it is closed under addition. That is, we have to show that if $\sum_{k=1}^{\infty} x_k^2$ and $\sum_{k=1}^{\infty} y_k^2$ both converge, then $\sum_{k=1}^{\infty}(x_k + y_k)^2$ converges. But the convergence of the first two series, gives us that $\sum_{k=1}^{\infty} x_k y_k$ converges by 4.a). Hence

$$\sum_{k=1}^{N}(x_k + y_k)^2 = \sum_{k=1}^{N}(x_k^2 + 2x_k y_k + y^2) \leq \sum_{k=1}^{\infty} x_k^2 + 2\sum_{k=1}^{\infty} x_k y_k + \sum_{k=1}^{\infty} y_k^2,$$

and we are through.

c) This is straight foreward, using the corresponding properties for series.

d) The main problem here is notational. We have to look at sequences of sequences!! So, we change notation slightly hopefully making thing a little clearer. A sequence is nothing but a function $\xi\colon \mathbb{N} \to \mathbb{R}$. The correspondence with the "old" notation is that $\xi_n = \xi(n)$. In the "new" notation, the norm is given by $\|\xi\|^2 = \sum_{k=1}^{\infty} \xi(k)^2$.

Now, a sequence in l_2 is a sequence $\{\xi_n\}$ of functions $\xi_n\colon \mathbb{N} \to \mathbb{R}$, and it is Cauchy if for every $\epsilon > 0$ there is an N such that

$$\|\xi_n - \xi_m\|^2 = \sum_{k=1}^{\infty} (\xi_n(k) - \xi_m(k))^2 \leq \epsilon^2 \qquad (✣)$$

whenever $n, m > N$. But this means that for each k we have $|\xi_n(k) - \xi_m(k)| < \epsilon$ for $n, m > N$. Hence each of the sequences $\{\xi_n(k)\}_{n=1}^{\infty}$ is a Cauchy sequence, and converges to some number $\xi(k)$. From ✣ we get for all M:

$$\sum_{k=1}^{M} (\xi_n(k) - \xi_m(k))^2 \leq \sum_{k=1}^{\infty} (\xi_n(k) - \xi_m(k))^2 \leq \epsilon^2$$

and first letting m tend to ∞, and then M tend to ∞, we arrive at:

$$\|\xi_n - \xi\| \leq \epsilon$$

for $n > N$, which shows that $\{\xi_n\}$ converges to ξ — which indeed is an element of l_2, since

$$\|\xi\| = \|\xi - \xi_n + \xi_n\| \leq \|\xi - \xi_n\| + \|\xi_n\| \leq \epsilon + \|\xi_n\|,$$

whenevere $n > N$. Hence $\|\xi\| < \infty$, and $\xi \in l_2$.

e) In our "new" notation the sequnce e_i is the function $e_i\colon \mathbb{N} \to \mathbb{R}$ given by $e_i(k) = 0$ if $i \neq k$ and $e_k(k) = 1$. Clearly each e_i is quadratically summable, having only one term different from zero. We want to show that $\xi = \sum_{i=1}^{\infty} \xi(i) e_i$. Which is reasonable: If we ignore convergence problems, the right side evaluates at k to $\xi(k)$ since $e_i(k) = 0$ if $i \neq k$ and $e_k(k) = 1$.

To check convergence we examine:

$$\left\| \xi - \sum_{i=1}^{n} \xi(i) e_i \right\|^2 = \sum_{i=n+1}^{\infty} \xi(i)^2.$$

This can be made arbitrarily small by choosing n sufficiently great since $\sum_{i=1}^{\infty} \xi(i)^2$ converges, and hence $\lim_{n\to\infty} \sum_{i=1}^{n} \xi(i)e_i = \xi$. ❑

PROBLEM : Take any $w \in V$. It can be written $w = \sum_{n=1}^{\infty} \langle\, w, v_i \,\rangle\, v_i$, and by Parseval's theorem we get

$$\sum_{n=1}^{\infty} \langle\, w, v_i \,\rangle^2 = \|w\|^2 \leq \infty$$

Hence the sequence $\alpha(i) = \langle\, w, v_i \,\rangle$ (in our "new" notation from problem 12) is quadratically summable and belongs to l_2. If w_k is a sequence in V, we get in this way a sequence α_k in l_2 with $\alpha_k(i) = \langle\, w_k, v_i \,\rangle$.

Assume now that w_k is Cauchy, then α_k is Cauchy in l_2; indeed:

$$\|\alpha_n - \alpha_m\|^2 = \sum_{i=0}^{\infty} (\alpha_n - \alpha_m)^2 = \|w_n - w_m\|^2$$

again by Parseval. But w_k being Cauchy, we can , given $\epsilon > 0$, find an N such that if $n, m > N$, the last term satisfies $\|w_n - w_m\|^2 < \epsilon^2$, and hence also $\|\alpha_n - \alpha_m\|^2 < \epsilon^2$.

We know that l_2 is complete, so let $\alpha = \lim_{n\to\infty} \alpha_k$. Then $\lim_{n\to\infty} w_k = \sum_{i=1}^{\infty} \alpha(i)v_i$. Indeed:

$$\left\| w_k - \sum_{i=1}^{\infty} \alpha(i)v_i \right\|^2 = \left\| \sum_{i=1}^{\infty} (\alpha_k(i) - \alpha(i))v_i \right\|^2 = \sum_{i=1}^{\infty} (\alpha_k(i) - \alpha(k))^2,$$

but this tends to zero since $\lim_{n\to\infty} \alpha_k = \alpha$ in l_2. ❑

PROBLEM 14:

a) This is straight forward, using well known properties of the integral

b) This is just a translation of the general Cauchy-Schwarz inequality to the present situation with

$$\int_a^b f(x)\overline{g(x)}\, dx = \langle\, f, g \,\rangle$$

and

$$\int_a^b |f(x)|^2\, dx = \|f\|^2 .$$

❑

PROBLEM 15:

a) The formal properties of an inner product follow from the corresponding properties of the integral. The main point to check is that integral is convergent, and to see that, we substitute $t = \cos u$. Then $dt = -\sin u du$. When $t = 1$ then $u = 0$, and $t = -1$ gives $u = \pi$. Hence

$$\int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}\, dt = \int_{0}^{\pi} f(\cos u)g(\cos u)\, du, \qquad (\star)$$

and the last integrand is bounded, hence the integral converges.
b) Since $T_n(\cos u) = \cos(n \arccos(\cos u)) = \cos(nu)$, u belonging to $[0, \pi]$, we get by $\star$ that

$$\langle\, T_n, T_m \,\rangle = \langle\, \cos nu, \cos mu \,\rangle_C$$

where $\langle\,,\rangle_C$ denotes the inner product in $C([0, \pi], \mathbb{R})$. And we can conclude, since we know that $\cos nu$ form an orthogonal set with respect to that product.
c) We use induction on n to show that $\cos nv$ is a polynomial in $\cos v$ of degree n, then putting $v = \arccos t$ and using $\cos \arccos t = t$ we are through. We use the now well known formulas:

$$\begin{aligned} 2 \sin\alpha \sin\beta &= \cos(\alpha - \beta) - \cos(\alpha + \beta) \\ 2 \sin\alpha \cos\beta &= \sin(\alpha + \beta) + \sin(\alpha - \beta) \end{aligned}$$

Putting $\alpha = t$ and $\beta = nt$ in the last one, we get

$$2 \sin t \cos nt = \sin(n + 1)t - \sin(n - 1)t.$$

We multiply by $2 \sin t$ and obtain:

$$\begin{aligned} 4 \sin t \cos nt &= 2 \sin t \sin(n + 1)t - 2 \sin t \sin(n - 1)t \\ &= \cos nt - \cos(n + 2)t - \cos(n - 2)t + \cos nt \\ &= \cos(n - 2) - \cos(n + 2) + 2 \cos nt, \end{aligned}$$

from wich it follows easyly by induction that $\cos(n+2)t$ is a polynomial of degree $n+2$. Indeed:

$$\cos(n+2)t = \cos(n-2) + 4t^2 \cos nt - 2\cos nt,$$

and by induction $\cos nt$ and $\cos(n-2)t$ are poly's of degree n and $n-2$ respectively. ❑