Ark8: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

Problem 1:

a) We compute

$$\int_0^a \frac{1}{a^2}(x-a)^2\,dx = \left.\frac{1}{3a^2}(x-a)^3\right|_0^a = \frac{a}{3}.$$

The L_2-norm is given by

$$\|T_{k,n}\|_2^2 = \int_{a_{k,n}-10^{-(n-1)}}^{a_{k,n}+10^{-(n-1)}} T_{k,n}^2\,dx = 2\cdot 10^{2(n-1)} \int_{a_{k,n}}^{a_{k,n}+10^{-(n-1)}} (x-a_{n,k}-10^{-(n-1)})^2\,dx$$

Following the hint, setting $a = 10^{-(n-1)}$ and substituting $y = x - a_{k,n}$ gives $dy = dx$ and new limits 0 and $a = 10^{-(n-1)}$, hence

$$\|T_{k,n}\|_2^2 = 2\frac{1}{a^2}\int_0^a (y-a)^2\,dy = 2\cdot\frac{a}{3} = 2\cdot 10^{-(n-1)}/3$$

Taking square roots we get $\|T_{k,n}\|_2 = \sqrt{6}/3\cdot 10^{-\frac{n-1}{2}}$.

b) Clearly $\lim_{n\to\infty}\|T_{k,n}\|_2 = \sqrt{6}/3\lim_{n\to\infty}10^{-\frac{n-1}{2}} = 0$; and when the indices of the sequence $T_{k,n}$ go to ∞ necessarily n also tends to ∞.

c) An easy computation gives $T_{k,n}(x) = -10^{n-1}|x-a_{k,n}|+1$. Hence if $|x-a_{k,n}| < 10^{-n}$ we get

$$T_{k,n}(x) = -10^{n-1}|x-a_{k,n}|+1 \geq -10^{n-1}\cdot 10^{-n}+1 = 9/10$$

d) One gets $k\cdot 10^{-n}$ by just keeping the n first decimals in the decimal expansion of x.

e) Given an N and an x, we can, for any $n > N$, find points $a_{k,n} = k\cdot 10^{-n}$ with $|x-k\cdot 10^{-n}| < 10^{-n}$, and hence $T_{k,n}(x)\geq 9/10$. On the other hand, by choosing $a_{k,n}$ far away from x, we get $T_{k,n}(x) = 0$; *e.g.*, if n is so big that $10^{-n} < \frac{x}{10}$, there are points of the form $a_{k,n}$ with distance to x greater than $10^{-(n-1)}$, and hence $T_{k,n}(x) = 0$. This shows that arbitrarily far out in the sequence there are k and n for wich $T_{k,n}(x) = 0$

and k and n for wich $T_{k,n}(x) \geq 9/10$, hence the sequence $\{T_{k,n}(x)\}$ does not converge. ❏

PROBLEM 2:

a) The sequence $\{\frac{1}{\sqrt{n}}\}$ decrease to zero when n tends to infinity. Hence $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ converges by Dirichlet's criterion (problem 1 on Ark7).

b) The function $f(x)$ can not be in L_2 since if it were, we would have by Parseval's formula

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n} = \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty,$$

which is not the case since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. ❏

PROBLEM 3:

a) We find:

$$\begin{aligned} c_n e^{inx} + c_{-n} e^{-inx} &= c_n \cos nx + c_{-n} \cos(-nx) + ic_n \sin nx + ic_{-n} \sin(-nx) \\ &= (c_n + c_{-n}) \cos nx + i(c_n - c_{-n}) \sin nx. \end{aligned}$$

b) If a_n and b_n are real, we get

$$\begin{aligned} c_n + c_{-n} &= \overline{c_n} + \overline{c_{-n}} \\ c_n - c_{-n} &= -\overline{c_n} + \overline{c_{-n}}, \end{aligned}$$

adding the equations gives $2c_n = 2\overline{c}_{-n}$, hence $c_n = \overline{c_{-n}}$. The other way is clear: $c_n + \overline{c_n}$ and $i(c_n - \overline{c_n})$ are obviously equal to their conjugates and hence real. It follows that $a_n = c_n + \overline{c}_n = 2\,\mathrm{Re}\; c_n$ and $b_n = i(c_n - \overline{c}_n) = -2\,\mathrm{Im}\; c_n$

c) If $f(x)$ is a real function, then the complex conjugate of $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$ is equal to $c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx$; so both a_n and b_n are real. Using the formulas $e^{-inx} + e^{inx} = 2\cos nx$ and $e^{-inx} - e^{inx} = 2i \sin nx$ we get the formulas in the problem.

d) If f is an odd function, we get since $\cos x$ is even, that $\int_{-\pi}^{0} f(x)\cos nx\,dx = -\int_{0}^{\pi} f(x)\cos nx\,dx$ (substitute $y = -x$), hence $a_n = 0$

e) If $f(x)$ is even, the $f(x)\sin nx$ is odd, since $\sin nx$ is odd. It follows that $b_n = 0$. ❑

PROBLEM 4: Assume first that $a \in [-\pi, \pi]$. Then we have

$$\begin{aligned}\int_a^{a+2\pi} f(x)\,dx &= \int_a^{\pi} f(x)\,dx + \int_\pi^{a+2\pi} f(x)\,dx = \\ &= \int_a^{\pi} f(x)\,dx + \int_{-\pi}^{a} f(x)\,dx = \\ &= \int_{-\pi}^{\pi} f(x)\,dx\end{aligned}$$

where we substituted $x - 2\pi$ for x in the last integral in the second formula.

If $a \notin [-\pi, \pi]$, we can write $a = a' + 2k\pi$ where k is an integer and $a' \in [-\pi, \pi]$. Substitute $x - 2k\pi$ for x to get

$$\int_a^{a+2\pi} f(x)\,dx = \int_{a'}^{a'+2\pi} f(x)\,dx.$$

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PROBLEM 5: We substitute $u = x - a$ in the integral defining the Fourier coefficient $c_n(f(x))$ and use problem 4 to obtain:

$$c_n(f(x)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi-a}^{\pi+a} f(u+a)e^{-in(u+a)}\,du = \\ \frac{1}{2\pi}e^{-ina}\int_{-\pi}^{\pi} f(u+a)e^{-inu}\,du = e^{-ina}c_n(f(x+a)).$$

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PROBLEM 6. 7

a) By using the formula in the hint with $\alpha = kx$ and $\beta = x/2$ we get by the "telescoping the property" of the sum:

$$\begin{aligned}\sum_{k=0}^{n} \sin kx &= \frac{1}{2\sin\frac{x}{2}}\sum_{k=1}^{n}(\cos(k-1/2)x - \cos(k+1/2)x) \\ &= \frac{\cos\frac{x}{2} - \cos\,(n+\frac{1}{2})x}{2\sin\frac{x}{2}}\end{aligned}$$

b) By using the formula in the hint with $\alpha = kx$ and $\beta = x/2$ we get by the "telescoping the property" of the sum:

$$\sum_{k=0}^{n} \cos kx = \frac{1}{2\sin\frac{x}{2}} \sum_{k=0}^{n} (\sin(k+1/2)x - \sin(k-1/2)x)$$
$$= \frac{\sin\frac{x}{2} + \sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

c) This is a slightly delicate point which we come back to in a later problem. The point is that both $\sin(n+\frac{1}{2})x$ and $\cos(n+\frac{1}{2})x$ oscillates strongly, in fact if x is irrational, the sets $\{\sin(n+\frac{1}{2})x : n \in \mathbb{N}\}$ an $\{\cos(n+\frac{1}{2})x : n \in \mathbb{N}\}$ are dense in $[-1,1]$, so the above formulas show that neither of the series $\sum_{k=0}^{\infty} \sin kx$ and $\sum_{k=0}^{\infty} \cos kx$ converges

PROBLEM 12:

a) We compute:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{izt}e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z-n)t}\,dt = \frac{1}{2\pi i(z-n)}\Big|_{-\pi}^{\pi} e^{i(z-n)t} =$$
$$\frac{1}{2\pi i(z-n)}\left(e^{i(z-n)\pi} - e^{-i(z-n)\pi}\right) = \frac{e^{n\pi}}{2\pi i(z-n)}\left(e^{iz\pi} - e^{-iz\pi}\right) = \frac{(-1)^n \sin \pi z}{\pi(z-n)}.$$

b) We compute the L_2-norm of e^{ixt} for $x \in \mathbb{R}$:

$$\left\|e^{ixt}\right\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt}e^{-ixt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt = 1.$$

The Parseval-identity then gives the result since $|c_n|^2 = \frac{\sin^2 \pi x}{\pi^2 (x-n)^2}$

c) To get the Fourier coefficients of e^x, we put $z = -i$ in the formula in a). That gives us

$$c_n = (-1)^n \frac{e^{i(-i\pi)} - e^{i(i\pi)}}{2\pi i(i-n)} = (-1)^{n+1}\frac{e^{\pi} - e^{-\pi}}{2\pi}\frac{(n+i)i}{n^2+1}$$

Hence

$$e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{(n+i)ie^{inx}}{n^2+1}.$$

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