Ark8: Exercises for MAT2400 — Fourier series

The exercises on this sheet cover the sections *4.7*, *4.8* and *4.9* of Tom's notes. They are ment for the groups on Thursday, Mars 22 and Friday, Mars 24. With the following distribution:

Thursday, Mars 22: No 5, 6, 8, 9, 10, 11
The rest for Friday.

Key words: L_2-convergence, Fourier series.

L_2 convergence

PROBLEM 1. The aim of this exercise is to give an example of a sequence of functions in the interval $[0, 1]$ that tend to zero in L_2-norm, but that do not converge pointwise at any point. We shall use the "tent" construction with which we are familiar, but are going to spread the tents all over the interval $[0, 1]$ by centering the tents at the points of a dense, but countable, subset.

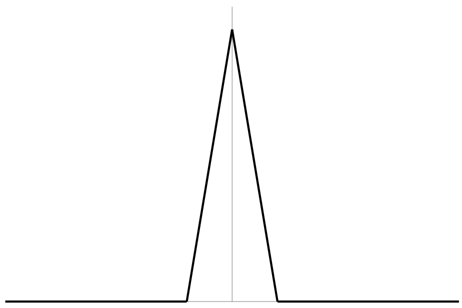
For any positive integers k and n with $k < 10^n$, we let $a_{k,n} = k \cdot 10^{-n}$. These numbers are just all the finite decimal expansions. One example being $0.12345 = 12345 \cdot 10^{-5}$.

We want to make a sequence out of the numbers $a_{k,n}$, so we order them by letting $a_{k,n}$ come before $a_{k',n'}$ if $n < n'$, and if $k < k'$ in case $n = n'$. Writing down the $a_{k,n}$ in this order, gives us the sequence we want, *i.e.,* one whose terms are the $a_{k,n}$'s. When we later in the exercise referes to the sequence $T_{k,n}$ it is understood that it is ordered in this way.

We define the following "tent"-function $T_{k,n}(x)$ centered at $a_{k,n}$ and with base $10^{-(n-1)}$ and height 1, by

$$T_{k,n}(x) = \begin{cases} 0 & \text{if } |x - a_{n,k}| > 10^{-(n-1)} \\ 10^{n-1}(x - a_{n,k} + 10^{-(n-1)}) & \text{if } -10^{-(n-1)} < x - a_{n,k} < 0 \\ -10^{n-1}(x - a_{n,k} - 10^{-(n-1)}) & \text{if } 0 < x - a_{n,k} < 10^{-(n-1)} \end{cases}$$

The graph of $T_{k,n}$ looks like this:



a) Show that $\int_0^a \frac{1}{a^2}(x-a)^2\,dx = \frac{a}{3}$ and use that to show that

$$\|T_{k,n}\|_2 = \sqrt{6}/3 \cdot 10^{-\frac{n-1}{2}}.$$

HINT: Set $a = 10^{-(n-1)}$ and substitute $x - a_{k,n}$ for x in the integral.

b) Show that $T_{k,n}$ tends to 0 in L_2-norm.

c) Show that if $|x - a_{k,n}| < 1/10^n$ then $T_{k,n}(x) \geq 9/10$.

d) Show that for any $x \in [0,1]$ and any n there is a k such that $|x - k \cdot 10^{-n}| < 10^{-n}$. HINT: Use the decimal expansion of x.

e) The sequence $T_{k,n}(x)$ does not converge pointwise in any point.

PROBLEM 2. This is to give an example of a series that converges pointwise but not in L_2.

a) Show that series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}} \qquad (✢)$$

converges for all x. HINT: Use Dirchlet's test from problem 1 on Ark7.

b) Show that the series in ✢ does not converge in L_2-norm — *i.e.*, the integral $\int_{-\pi}^{\pi} |f(x)|^2\,dx$ can not be finite. HINT: Parseval's identity.

Fourier coefficients

PROBLEM 3. In this exercise let c_n be a sequence of complex numbers where $n \in \mathbb{Z}$ (*i.e.*, also negative integers allowed). Let a_n and b_n for $= 0, 1, 2, \ldots,$ be given as $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$

a) Show that we have the following equality of trigonometric polynomials

$$\sum_{n=-N}^{N} c_n e^{inx} = a_0/2 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx).$$

b) Show further that numbers a_n and b_n all are real if and only if $c_{-n} = \overline{c_n}$ for all n. And in that case, $a_n = 2\,\mathrm{Re}\; c_n$ and $b_n = -2\,\mathrm{Im}\; c_n$.

Let now c_n be the Fourier coefficients of a function $f(x)$ defined for $x \in [-\pi, \pi]$.

c) If $f(x)$ is a *real* function, show that its Fourier series is given by

$$a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$. In particular, a_0 is the average of f over $[-\pi, \pi]$.

d) Show that f is *real* and *odd*, *i.e.,* satisfies $f(-x) = -f(x)$ for $x \in [-\pi, \pi]$, then $c_{-n} = -c_n$ and $a_n = 0$ for all n. Conclude thay if f is real and odd, its Fourier series is of the form

$$\sum_{n=1}^{\infty} b_n \sin nx,$$

where $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$.

e) Show that if $f(x)$ is *real* and *even*, *i.e.,* satisfies $f(-x) = f(x)$ for $x \in [\pi.\pi]$, then $c_{-n} = c_n$ and $b_n = 0$ for all n. Conclude that if f is real and even, its Fourier series is of the form

$$a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx.$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$.

PROBLEM 4. Assume that $f(x)$ is a function defined on $\mathbb{R}$ which is periodic with period 2π. Show that for any $a \in \mathbb{R}$ we have

$$\int_a^{a+2\pi} f(x)\, dx = \int_0^{2\pi} f(x)\, dx$$

i.e., the integrals of f over intervals of length 2π are all equal.

PROBLEM 5. Let $f(x)$ be defined in $\mathbb{R}$, and assume that f is periodic with period 2π. Let a a real constant. Let $c_n(f(x))$ denote the n-th Fourier coefficient of f. Show that $c_n(f(x-a)) = e^{-ian}c_n(f(x))$.

PROBLEM 6. Assume that $a \in \mathbb{R}^+$ and that $f(x)$ is a function on $\mathbb{R}$ which is periodic with period $2a$.

a) Show that if we let $g(x) = f(\frac{a}{\pi}x)$, then g is periodic with period 2π.
b) Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-\frac{n\pi}{a}x}\,dx.$$

PROBLEM 7. *(Basically Tom's notes 4,7, Problem 7 d) (page 117)).*
a) Show that

$$\sum_{k=0}^{n} \sin kx = \frac{\cos\frac{x}{2} - \cos{(n+\frac{1}{2})x}}{2\sin\frac{x}{2}}$$

whenever x is not an even multiple of π. HINT: Use the formula $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha+\beta)$ and a "telescoping" property.
b) Show that

$$\sum_{k=0}^{n} \cos kx = \frac{\sin\frac{x}{2} + \sin{(n+\frac{1}{2})x}}{2\sin\frac{x}{2}}$$

whenever x is not an even multiple of π. HINT: Use the formula $2\cos\alpha\sin\beta = \sin(\alpha + \beta) - \sin(\alpha-\beta)$ and a "telescoping" property.
c) Does the series $\sum_{k=0}^{\infty}\sin kx$ and $\sum_{k=0}^{\infty}\cos kx$ converge?

Fourier series

PROBLEM 8. *(Tom's notes 4.7, Problem 5 (page 117)).* Find the Fourier series of $\sin\frac{x}{2}$ on $[-\pi,\pi]$.

PROBLEM 9. Let $\phi(x) = \frac{x}{|x|}$ if $x \neq 0$ and let $f(0) = 0$ (*i.e.*, The function ϕ satisfies $\phi(x) = -1$ if $x < 0$ and $\phi(x) = 1$ if $x > 0$. It is often called the "square puls" when continued to the entire real line by periodicity).
a) Show that the Fourier series of ϕ is given as

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin{(2n+1)x}}{2n+1}.$$

b) Conclude that

$$\frac{\pi^2}{16} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

PROBLEM 10. Let $f(x)$ be the function definet for all $x \in \mathbb{R}$ which is periodic with period 2π and which satisfies $f(x) = 1$ for $x \in [-h, h]$ and $f(x) = 0$ if $x \notin [-h, h]$ where $h \in (0, \pi)$.

a) Make a sketch of det graph of f.

b) Show that the Fourier series of f is

$$\frac{h}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nh}{n} \cos nx.$$

PROBLEM 11. Let $f(x) = |x|$ for $x \in [-\pi, \pi]$.

a) Show that the Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos{(2n+1)x}}{(2n+1)^2}.$$

b) Determine $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}$.

PROBLEM 12.

a) Assume that $z \in \mathbb{C}$ is not an integer. Show that the Fourier coefficients of the function $f(t) = e^{izt}$, $t \in [-\pi, \pi]$ are given by

$$c_n = \frac{(-1)^n \sin \pi z}{\pi(z-n)}.$$

b) Use Parseval's identity to show that

$$\frac{\pi^2}{\sin^2 \pi x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2}.$$

for x real, not an integer.

c) Find the Fourier series of e^x on $[-\pi, \pi]$.

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