UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT2400 — Analysis 1.

Day of examination: August ??, 2011.

Examination hours: ??–??

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (1, 2, 3a, 3b etc.) count 10 points each.

Problem 1

The function $f : [-\pi, \pi] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [-\pi, 0) \\ 1 & \text{if } x \in [0, \pi] \end{cases}$$

a) Show that if $n \in \mathbb{Z}$, then

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \begin{cases} \pi & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even and different from } 0 \\ -\frac{2i}{n} & \text{if } n \text{ is odd} \end{cases}$$

b) Find the Fourier series of f. To which function does this series converge pointwise? Is the convergence uniform?

c) Show that the Fourier series can be written

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2\sin\big((2k+1)x\big)}{\pi(2k+1)}$$

Problem 2

If (X, d) is a metric space, and $\{x_n\}$ is a sequence in X, we call $a \in X$ a *limit point* for $\{x_n\}$ if there is a subsequence of $\{x_n\}$ converging to a. Show that a is a limit point for $\{x_n\}$ if and only if all balls $B(a; r)$, $r > 0$, centered at a contain infinitely many terms of the sequence.

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Problem 3

In this problem μ is the Lebesgue measure on $\mathbb{R}^d$, and B_n is the ball

$$B_n = \{x \in \mathbb{R}^d \,:\, \|x\| \leq n\}$$

a) Show that if $f : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative, measurable function, then

$$\lim_{n\to\infty} \int_{B_n} f \, d\mu = \int f \, d\mu$$

b) Show that if $g : \mathbb{R}^d \to \mathbb{R}$ is an integrable function, then

$$\lim_{n\to\infty} \int_{B_n} g \, d\mu = \int g \, d\mu$$

Problem 4

In this problem X is the set of all functions

$$f : \mathbb{N} \to \mathbb{R}$$

such that the limit $\lim_{i\to\infty} f(i)$ exists (the limit should be a number; we do not accept $\pm\infty$ as limit values).

a) Show that

   (i) If $f \in X$ and $c \in \mathbb{R}$, then $cf \in X$.

   (ii) If $f, g \in X$, then $f + g \in X$.

The result in a) tells us that X is a vector space. You may use this freely in what follows.

b) Show that $\sup\{|f(i)| \,:\, i \in \mathbb{N}\}$ is finite for all $f \in X$.

c) Show that

$$\|f\| = \sup\{|f(i)| \,:\, i \in \mathbb{N}\}$$

   is a norm on X.

d) Show that X is complete.

Problem 5

In this problem (X, d_X) and (Y, d_Y) are two compact metric spaces, and $f : X \to Y$ is an invertible, continuous function. Show that the inverse function $g : Y \to X$ is continuous.

THE END