MATH2400 Exam August 2013. Solution

Poblem 1: Since f is bounded, there is a constant $M \in \mathbb{R}$ such that $|f(t)| \leq M$ for all t. Hence if $y \leq x$,

$$|F(x) - F(y)| = |\int_0^x f(t)\,dt - \int_0^y f(t)\,dt| = |\int_y^x f(t)\,dt| \leq$$

$$\leq \int_y^x |f(t)|\,dt \leq \int_y^x M\,dt = M(x-y)$$

Given $\epsilon > 0$, we may choose $\delta = \frac{\epsilon}{M}$ and have $|F(x) - F(y)| < \epsilon$ whenever $|x - y| < \delta$. This shows that F is uniformly continuous.

The problem may also be solved using the Mean Value Thorem.

Problem 2: Let $\epsilon > 0$. Since $\{\mathbf{x}_n\}$ is a Cauchy sequence, there is a number $N_1 \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}_m\| < \frac{\epsilon}{2}$ when $n, m \geq N_1$. Similarly, since $\{\mathbf{y}_n\}$ is a Cauchy sequence, there is a number $N_2 \in \mathbb{N}$ such that $\|\mathbf{y}_n - \mathbf{y}_m\| < \frac{\epsilon}{2}$ when $n, m \geq N_2$. Choose $N = \max\{N_1, N_2\}$, then if $n, m \geq N$

$$\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x}_m + \mathbf{y}_m)\| = \|(\mathbf{x}_n - \mathbf{x}_m) + (\mathbf{y}_n - \mathbf{y}_m)\| \leq$$

$$\leq \|\mathbf{x}_n - \mathbf{x}_m\| + \|\mathbf{y}_n - \mathbf{y}_m\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that $\{\mathbf{x}_n + \mathbf{y}_n\}$ is a Cauchy sequence.

Problem 3: a) Let $\epsilon > 0$; we must show that there is an $N \in \mathbb{N}$ such that when $n \geq N$, $|f(g_n(x)) - f(g(x))| < \epsilon$ for all $x \in \mathbb{R}$. Since f is uniformly continuous, there is a $\delta > 0$ such that $|f(u) - f(v)| < \epsilon$ whenever $|u - v| < \delta$, and since $\{g_n\}$ converges uniformly to g, there is an $N \in \mathbb{N}$ such that when $n \geq N$, $|g_n(x) - g(x)| < \delta$ for all x. This means that for $n \geq N$, we have $|f(g_n(x)) - f(g(x))| < \epsilon$ for all $x \in \mathbb{R}$ just as we wanted.

b) Since $|g_n(x) - g(x)| = \frac{1}{n}$, the sequence $\{g_n\}$ clearly converges uniformly to g (given $\epsilon > 0$, just choose $N > \frac{1}{\epsilon}$ to get $|g_n(x) - g(x)| = \epsilon$ for all x and all $n \geq N$).

On the other hand, since

$$|f(g_n(x)) - f(g(x))| = |(x + \frac{1}{n})^2 - x^2| = |\frac{2x}{n} + \frac{1}{n^2}|$$

we see that no matter how big n is, we can get $|f(g_n(x)) - f(g(x))|$ as big as we want by choosing x appropriately, and hence $f(g_n(x))$ does *not* converge uniformly to $f(g(x))$. The reason this do not contradict part a), is that $f(x) = x^2$ is not *uniformly* continuous as required there.

Problem 4: a) Assume that $\{\mathbf{x}\}_n$ converges to $\mathbf{x}$ in norm. By Cauchy-Schwarz' inequality

$$|\langle \mathbf{x} - \mathbf{x}_n, \mathbf{a} \rangle| \leq \|\mathbf{x} - \mathbf{x}_n\| \|\mathbf{a}\| \to 0$$

for all $\mathbf{a} \in X$, and hence $\{\mathbf{x}\}_n$ converges weakly to $\mathbf{x}$.

b) Any $\mathbf{a} \in X$ can be written as a sum $\mathbf{a} = \sum_{i=0}^{\infty} \alpha_i \mathbf{e}_i$, where $\|\mathbf{a}\|^2 = \sum_{i=1}^{\infty} \alpha_i^2$. Since the series converges, $\lim_{i\to\infty} \alpha_i = 0$, and we get

$$\langle \mathbf{e}_n - \mathbf{0}, \mathbf{a} \rangle = \langle \mathbf{e}_n, \sum_{i=0}^{\infty} \alpha_i \mathbf{e}_i \rangle = \sum_{i=1}^{\infty} \alpha_i \langle \mathbf{e}_n, \mathbf{e}_i \rangle = \alpha_n \to 0$$

which shows that $\{\mathbf{x}\}_n$ converges weakly to $\mathbf{0}$. As $\|\mathbf{e}_n - \mathbf{0}\| = \|\mathbf{e}_n\| = 1$, the sequence does not converge to $\mathbf{0}$ in norm.

Problem 5: a) *All* closed balls in $\mathbb{R}^m$ are compact since they are closed and bounded.

b) First note that if $n \neq m$,

$$\|\mathbf{e}_n - \mathbf{e}_m\|^2 = \langle \mathbf{e}_n - \mathbf{e}_m, \mathbf{e}_n - \mathbf{e}_m \rangle = \langle \mathbf{e}_n, \mathbf{e}_n \rangle - 2\langle \mathbf{e}_n, \mathbf{e}_m \rangle + \langle \mathbf{e}_m, \mathbf{e}_m \rangle = 1 - 2 \cdot 0 + 1 = 2$$

and hence $\|\mathbf{e}_n - \mathbf{e}_m\| = \sqrt{2}$.

Let $\overline{\mathrm{B}}(\mathbf{0}, r)$ be a closed ball around the origin. The sequence $\{r\mathbf{e}_n\}$ lies in this ball, but no subsequence of it can be a Cauchy sequence since $\|r\mathbf{e}_n - r\mathbf{e}_m\| = r\|\mathbf{e}_n - \mathbf{e}_m\| = r\sqrt{2}$ whenever $n \neq m$. Hence $\{r\mathbf{e}_n\}$ does not have a convergent subsequence, and $\overline{\mathrm{B}}(\mathbf{0}, r)$ is *not* compact for any $r > 0$.

Problem 6: a) Since $\|f_n - 0\|_1 = \int |f_n - 0| \, d\mu = \int f_n \, d\mu = 1$, $\{f_n\}$ does not converge to 0 in L^1-norm. On the other hand, if $\epsilon > 0$,

$$\mu(\{x \in X : |f(x) - 0| \geq \epsilon\}) \leq \frac{1}{n} \to 0 \qquad \text{as } n \to \infty$$

which shows that $\{f_n\}$ converges to 0 in measure.

b) The sequence consists of simple functions. The first element is the simple function of the entire interval [0,1], then come the simple functions of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, followed by the simple functions of the intervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. Continuing in this manner, we get a sequence that clearly converges to 0 in measure (as the intervals get shorter and shorter), but which doesn't converge at any point x (since x will be contained in infinitely many of the intervals).

c) We argue contrapositively: Assume that $\{f_n\}$ does *not* converge to f in measure. This means that there is an $\epsilon > 0$ such that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$$

does *not* go 0, and hence there is an $\alpha > 0$ such that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \geq \alpha$$

for infinitely many n. For these n,

$$\|f_n - f\|_1 = \int |f_n - f| \, d\mu \geq \epsilon\alpha,$$

and hence $\{f_n\}$ cannot converge to f in L^1-norm.

d) Assume that $\epsilon > 0$, and let

$$B_N = \{x \in X : \text{there exists } n \geq N \text{ such that } |f_n(x) - f(x)| \geq \epsilon\}) =$$

$$= \bigcup_{n \geq N} \{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$$

The sets B_N are measurable, the sequence $\{B_N\}$ is decreasing, and since $\{f_n\}$ converges to f almost everywhere, $\mu(\bigcap_{N \in \mathbb{N}} B_N) = 0$. By continuity of measure, $\mu(B_N) \to 0$, and hence

$$\mu(\{x \in X : |f_N(x) - f(x)| \geq \epsilon\}) \to 0$$

since $\{x \in X : |f_N(x) - f(x)| \geq \epsilon\} \subset B_N$. This proves convergence in measure.