UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

New/Deferred examination in: MAT2400 — Real analysis.
Day of examination: Friday 16. august 2013.
Examination hours: 09.00 – 13.00
This problem set consists of 2 pages.
Appendices: None.
Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (1, 2, 3, 4a, 4b etc.) count 10 points each.

Problem 1

Show that if $f : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function, then $F(x) = \int_0^x f(t)\,dt$ is uniformly continuous on $\mathbb{R}$.

Problem 2

Assume that $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are two Cauchy sequences in a normed space $(X, \|\cdot\|)$. Show that $\{\mathbf{x}_n + \mathbf{y}_n\}$ is also a Cauchy sequence.

Problem 3

a) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function and that $\{g_n\}$ is a sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ converging uniformly to $g : \mathbb{R} \to \mathbb{R}$. Show that the functions $h_n(x) = f(g_n(x))$ converge uniformly to $h(x) = f(g(x))$.

b) Let $f(x) = x^2$ and $g_n(x) = x + \frac{1}{n}$. Show that the functions $g_n(x)$ converge uniformly to $g(x) = x$, but that the functions $f(g_n(x))$ do *not* converge uniformly to $f(g(x))$. Why doesn't this contradict part a)?

Problem 4

A metric space (X, d) is called *locally compact* if there around every point $x \in X$ is a closed ball $\overline{\mathrm{B}}(x; r)$, $r > 0$, which is compact.

a) Show that $\mathbb{R}^m$ (with the usual metric) is locally compact.

b) Assume that $(X, \langle\cdot,\cdot\rangle)$ is an inner product over $\mathbb{R}$ with orthonormal basis $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$. Show that X is *not* locally compact. It may be advantageous first to show that $\|\mathbf{e}_n - \mathbf{e}_m\| = \sqrt{2}$ when $n \neq m$.

(Continued on page 2.)

Problem 5

In this problem $(X, \langle\cdot,\cdot\rangle)$ is an inner product space over $\mathbb{R}$, and $\|\cdot\|$ is the norm generated by $\langle\cdot,\cdot\rangle$. If $\{\mathbf{x}\}_{n\in\mathbb{N}}$ is a sequence of elements in X, we say that $\{\mathbf{x}_n\}$ *converges to* $\mathbf{x} \in X$ *in norm* if $\lim_{n\to\infty} \|\mathbf{x} - \mathbf{x}_n\| = 0$ (this is just the usual kind of convergence in a normed space). We say that the sequence $\{\mathbf{x}_n\}$ *converges weakly to* $\mathbf{x} \in X$ if $\lim_{n\to\infty}\langle \mathbf{x} - \mathbf{x}_n, \mathbf{a}\rangle = 0$ for all $\mathbf{a} \in X$.

a) Show that if $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$ in norm, then $\{\mathbf{x}_n\}$ also converges weakly to $\mathbf{x}$.

b) Assume that $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for X. Show that the sequence $\{\mathbf{e}\}_{n\in\mathbb{N}}$ converges weakly to $\mathbf{0}$, but that it doesn't converge to $\mathbf{0}$ in norm.

Problem 6

In this problem μ is the Lebesgue measure on the interval $[0, 1]$, and all functions are measurable functions from $[0, 1]$ to $\mathbb{R}$. We say that the sequence $\{f_n\}$ *converges to* f *in measure* if

$$\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) \to 0$$

for all $\epsilon > 0$.

a) Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & \text{if } x \in [0, \frac{1}{n}] \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that the sequence $\{f_n\}$ converges to 0 in measure, but that it does not converge to 0 in L^1-norm.

b) Give an example of a sequence $\{f_n\}$ which converges to 0 in measure, but such that $\{f_n(x)\}$ does not converge to 0 at *any* point $x \in [0, 1]$.

c) Show that if $\{f_n\}$ converges to f in L^1-norm, then $\{f_n\}$ converges to f in measure.

d) Show that if $\{f_n\}$ converges to f almost everywhere, then $\{f_n\}$ converges to f in measure.

THE END