MAT2400: Solution to the exam 2. June, 2014

Problem 1: a) We shall prove that $\{f_n\}$ converges pointwise to 0. For $x = 0$ this is obvious as $f_n(0) = 0$ for all n. For $x \neq 0$, we note that

$$f_n(x) = n^2x^2e^{-nx} = \frac{n^2x^2}{e^{nx}}$$

which shows that $f_n(x) \to 0$ as the exponential e^{nx} grows faster than the polynomial n^2. An alternative justification is to note that by L'Hôpital's rule (remember to differentiate with respect to n):

$$\lim_{n\to\infty} \frac{n^2x^2}{e^{nx}} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2nx^2}{xe^{nx}} = \lim_{n\to\infty} \frac{2nx}{e^{nx}} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2x}{xe^{nx}} = 0$$

b) The convergence is *not* uniform. The simplest way to see this, is probably to note that $f_n(1/n) = e^{-1}$, which means that the distance from f_n to 0 is always at least e^{-1}.

If you don't see this, the standard method is to compute the maximal distance $|f_n(x) - 0|$ between f_n and the limit function 0 as x varies. Since $|f_n(x) - 0| = f_n(x)$, we differentiate $f_n(x)$ to find the maximum:

$$f_n'(x) = 2n^2xe^{-nx} - n^3x^2e^{-nx} = n^2xe^{-nx}(2 - nx)$$

This implies that the maximum is at $x = \frac{2}{n}$ and that the maximal value is

$$f_n(2/n) = n^2\left(\frac{2}{n}\right)^2 e^{-n\cdot\frac{2}{n}} = 4e^{-2}$$

Since the maximum value does *not* go to 0, the convergence is *not* uniform.

Problem 2: Let $\{y_n\}$ be a Cauchy sequence in X_2; we must prove that it converges. Since ϕ is a bijection, there are points x_n in X_1 such that $\phi(x_n) = y_n$, and since ϕ is an isometry, $d_1(x_n, x_m) = d_2(y_n, y_m)$. This means that $\{x_n\}$ is also a Cauchy sequence, and since X_1 is complete, it must converge to a point $x \in X_1$. Let $y = \phi(x)$. Then

$$d_2(y, y_n) = d_2(\phi(x), \phi(x_n)) = d_1(x, x_n) \to 0 \quad \text{as } n \to \infty$$

and hence $\{y_n\}$ converges to y.

Problem 3: If $\mu(X) < \infty$, the constant function $g = M$ is integrable since $\int g\, d\mu = M\mu(X) < \infty$. Hence we can use Lebesgue's Dominated Convergence Theorem with g as the dominating function to get

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu$$

To find a counterexample (there are many!), let $(\mathbb{R}, \mathcal{A}, \mu)$ be the usual Lebesgue measure space, and choose

$$f_n(x) = \begin{cases} 1 & \text{if } n \leq x < n+1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Note that $\int f_n\, d\mu = 1$ for all n, but that since $\{f_n\}$ converges pointwise to $f = 0$, we have $\int f\, d\mu = 0$. This means that $\int f_n\, d\mu$ does *not* converge to $\int f\, d\mu$ in this case.

Problem 4: a) Pick a set $A \in \mathcal{D}$ (such a set exists since $\mathcal{D}$ is nonempty.) By (i), A^c is in $\mathcal{D}$, and by (ii) so is $\emptyset = A \cap A^c$.

b) We use induction on n. For $n = 2$ the statement holds as it is identical to condition (ii). We proceed by induction and assume that the statement holds for $n = k$, i.e. $A_1 \cap A_2 \cap \ldots \cap A_k \in \mathcal{D}$. By (ii), we then get

$$A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1} = (A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1} \in \mathcal{D}$$

which shows that the statement holds for $k+1$. By induction, it holds for all n.

c) We use b) and one of De Morgan's laws: Since $A_1, A_2, \ldots, A_n \in \mathcal{D}$, we have $A_1^c, A_2^c, \ldots, A_n^c \in \mathcal{D}$ by (i). By b), $A_1^c \cap A_2^c \cap \ldots \cap A_n^c \in \mathcal{D}$, and by (i) and De Morgan's law

$$A_1 \cup A_2 \cup \ldots \cup A_n = (A_1^c \cap A_2^c \cap \ldots \cap A_n^c)^c \in \mathcal{D}$$

d) From a) we know that $\emptyset \in \mathcal{D}$, and by (i) we know that if $A \in \mathcal{D}$, then $A^c \in \mathcal{D}$. Hence it suffices to show that if $\{B_n\}$ is a sequence of sets in $\mathcal{D}$, then $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{D}$. The obvious plan is to use condition (iii), but we first need to turn $\{B_n\}$ into a disjoint sequence $\{A_n\}$.

We define

$$A_1 = B_1, A_2 = B_2 \cap B_1^c, \ldots, A_n = B_n \cap (B_1 \cup B_2 \cup \ldots \cup B_{n-1})^c, \ldots$$

and observe that by (i), b) and c), all the A_n's are in $\mathcal{D}$. Since they are also disjoint by construction, we can use (iii) to conclude that

$$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$$

Problem 5: a) We have to check that d satisfies the three properties of a metric.

(i) Positivity: We have $d(x, y) = d_1(x, y) + d_2(x, y) \geq 0$ with equality if and only if $x = y$ since this holds for d_1 and d_2.

(ii) Symmetry: $d(x, y) = d_1(x, y) + d_2(x, y) = d_1(y, x) + d_2(y, x) = d(y, x)$ since d_1 and d_2 are symmetric.

(iii) Triangle inequality: For all points $x, y, z \in X$, we have

$$d(x, y) = d_1(x, y) + d_2(x, y) \leq d_1(x, z) + d_1(z, y) + d_2(x, z) + d_2(z, y)$$

$$= d_1(x, z) + d_2(x, z) + d_1(z, y) + d_2(z, y) = d(x, z) + d(z, y)$$

since d_1 and d_2 satisfies the triangle inequality.

b) We must show that any sequence $\{x_n\}$ in C has a subsequence that converges to a point $x \in C$ in the d-metric. Since C is compact in the d_1-metric, we know that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $x \in C$ in the d_1-metric, and since C is compact in the d_2-metric, we know that $\{x_{n_k}\}$ has a subsequence $\{x_{n_{k_j}}\}$ converging to a point $y \in C$ in the d_2-metric. As a subsequence of $\{x_{n_k}\}$, $\{x_{n_{k_j}}\}$ also has to converge to x in the d_1-metric, and since the two metrics are compatible, this mean that $x = y$. Hence

$$d(x_{n_{k_j}}, x) = d_1(x_{n_{k_j}}, x) + d_2(x_{n_{k_j}}, x) \to 0 \quad \text{as } j \to \infty$$

This shows that C is compact with respect to d.

Problem 6: a) For the first part, note that if we use the definition of convergence with $\epsilon = 1$, we get an $N \in \mathbb{N}$ such that if $n \geq N$, then

$|x_n - a| < 1$. It follows that $|x_n| < |a| + 1$ for all $n \geq N$. This means that $\{x_n\}$ is bounded by

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{n-1}|, |a| + 1\}$$

For the second part, assume that $\epsilon > 0$. Since $\{x_n\}$ converges to a, there is an $N \in \mathbb{N}$ such that $|a - x_n| < \frac{\epsilon}{2}$ when $n \geq N$. By what we have already proved, $\{x_n\}$ is bounded by an $M \in \mathbb{R}$, and hence for any $n \geq N$:

$$|a - y_n| = \left| a - \frac{x_1 + x_2 + \cdot + x_n}{n} \right| = \left| \sum_{i=1}^{n} \frac{a - x_i}{n} \right| \leq \sum_{i=1}^{n} \left| \frac{a - x_i}{n} \right| \leq$$

$$\leq \sum_{i=1}^{N} \left| \frac{a - x_i}{n} \right| + \sum_{i=N+1}^{n} \left| \frac{a - x_i}{n} \right| \leq N \frac{|a| + M}{n} + \frac{n - N}{n} \cdot \frac{\epsilon}{2}$$

The last term is always less than $\frac{\epsilon}{2}$, and the next to last term we can get less than $\frac{\epsilon}{2}$ by choosing n large enough. This means that we can get $|y_n - a| < \epsilon$ by choosing n sufficiently large, and hence $\{y_n\}$ converges to a.